IB Math AA SL — Topic 1

Exponentials & Logs

Introduction to Logarithms · Laws of Logarithms · Exponential Equations

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:________________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Definition:** $\log_a x = y \iff a^y = x \quad (a > 0,\ a \neq 1,\ x > 0)$
- **Laws of logarithms:** $\log_a(xy) = \log_a x + \log_a y$; $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$; $\log_a(x^n) = n\log_a x$
- **Change of base:** $\log_a x = \frac{\ln x}{\ln a} = \frac{\log x}{\log a}$
- **Special values:** $\log_a 1 = 0$; $\log_a a = 1$; $\ln e = 1$; $\ln 1 = 0$
- **Inverse relationship:** $a^{\log_a x} = x$; $\log_a(a^x) = x$
- **Solving exponential equations:** $a^x = b \Rightarrow x = \frac{\ln b}{\ln a}$ (exact form preferred on Paper 1)
- **Hidden quadratic substitution:** let $m = a^x$, solve $pm^2 + qm + r = 0$, return via $x = \frac{\ln m}{\ln a}$, reject $m \leq 0$

GDC Tips

- **Evaluating logarithms:** use `log()` (base 10) or `ln()` keys directly; for other bases type `log(x)/log(a)` or use the `logBASE` template in the Math menu.
- **Solving exponential/log equations:** enter both sides as Y_1 and Y_2, use `Calc` $\to$ `Intersect`; or rewrite as $f(x) = 0$ and use `Calc` $\to$ `Zero`. Quote x to 3 s.f. unless an exact form is required.
- **Checking exact ln-form answers:** evaluate your exact expression (e.g. $\ln 5/\ln 3$) on the home screen and confirm it matches the decimal from the graph screen.
- **Storing intermediate values:** store a computed ln value with `STO`$\to$ before substituting back, to avoid compounding rounding errors across parts.
- **Sketching exponential/log graphs:** set a window such as $[-2, 5] \times [-3, 8]$; use `Trace` to confirm the y-intercept and asymptote behaviour; zoom in with `Zoom` $\to$ `ZBox` near intersections.
- **Growth/decay threshold time:** after finding k exactly, store it and use `Calc` $\to$ `Zero` on $f(t) -$ threshold $= 0$ to find t numerically, then give the answer to 3 s.f. (or nearest integer if a whole-number context).

Common Mistakes to Avoid

1. **Forgetting to check argument positivity.** Every logarithm $\log_a f(x)$ requires $f(x) > 0$. After solving a log equation, substitute each candidate back and reject any value that makes an argument zero or negative. Omitting this check loses the domain-rejection A1.
2. **Rejecting the negative root of the substitution quadratic too late (or not at all).** When $m = a^x$ is introduced, m must be strictly positive. Write "$m = -2$ rejected since $a^x > 0$" explicitly; a bare cross through the root scores nothing.
3. **Incorrectly applying the product/quotient law.** $\log(a+b) \neq \log a + \log b$ and $\log(a-b) \neq \log a - \log b$. The addition/subtraction laws apply only to $\log(a \times b)$ and $\log(a \div b)$.
4. **Leaving an inexact answer on a non-calculator question.** Paper 1 requires exact forms such as $\dfrac{\ln 5}{\ln 2}$ or $\ln 12$. Writing a decimal approximation (e.g. 2.32) loses the final A1 even if the method is correct.
5. **Confusing "solve" with finding only one solution.** When a trig or exponential equation is solved on a given domain, all solutions must be found. For exponential equations with a substitution quadratic, both roots of the quadratic must be considered before rejection; ignoring a root loses marks.
6. **Circular reasoning in a "show that" (AG) part.** Working towards a printed target must proceed forward from the given information. Starting from the target and working backwards, or verifying numerically, receives no credit. Ensure every step is a logical consequence of the previous one, with the final target line following clearly from the line above.

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Section A — Easy

EASY

1. Write down the value of $\log_3 81$. *Calculator not allowed* **[3 marks]**

2. Write down the value of $\log_5 1$. *Calculator allowed* **[3 marks]**

3. Find the exact value of x such that $2^x = 32$, where $x \in \mathbb{R}$. *Calculator allowed* **[3 marks]**

4. Express $\log_4 64$ as an integer. *Calculator allowed* **[3 marks]**

5. Given that $\ln x = 3$, find the exact value of x, where $x \in \mathbb{R}^+$. *Calculator not allowed* **[3 marks]**

6. The table below shows selected values of a function $f(x) = \log_{10} x$ for $x > 0$.

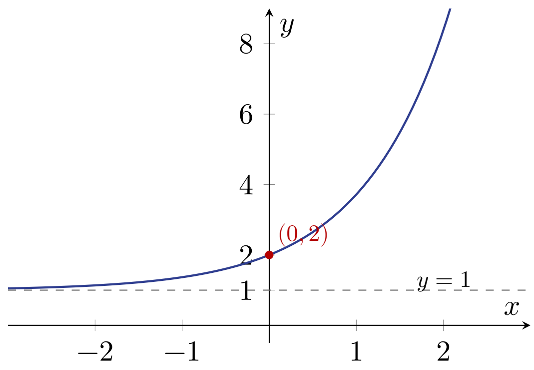
x	10	100	1000	10000
$f(x)$	1	a	b	4

Find the values of a and b. *Calculator allowed* **[3 marks]**

7. Write $\log_2 8 + \log_2 4$ as a single logarithm, and hence find its exact value. *Calculator not allowed* **[3 marks]**

8. Solve the equation $3^x = 20$ for $x \in \mathbb{R}$, giving your answer correct to 3 significant figures. *Calculator allowed* **[3 marks]**

9. The sketch below shows the graph of $f(x) = e^x + 1$ for $x \in \mathbb{R}$.



Write down the equation of the horizontal asymptote of the graph of f. *Calculator allowed* **[3 marks]**

10. Express $\log_6 12 + \log_6 3$ as a single logarithm, and hence find its exact value. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. Let $\log_3 p = 5$ and $\log_3 q = 2$, where $p, q \in \mathbb{R}^+$. Find the value of $\log_3 \left(\frac{p\sqrt{q}}{9} \right)$. *Calculator not allowed* **[4 marks]**

12. Solve the equation $2\ln(x-1) - \ln(x+5) = \ln 2$, where $x \in \mathbb{R}$ and $x > 1$. *Calculator not allowed* **[4 marks]**

13. The number of bacteria in a culture is modelled by $B(t) = 400\,e^{kt}$, where t is measured in hours and k is a positive constant. It is given that $B(3) = 1600$.

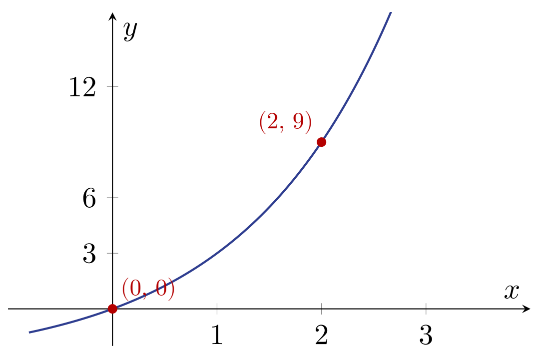
(a) Find the exact value of k. **[2 marks]**

(b) Find the value of t when $B(t) = 10000$. Give your answer correct to 3 significant figures. **[2 marks]**

Calculator allowed **[4 marks]**

14. Solve the equation $4^x - 6 \cdot 2^x + 8 = 0$, where $x \in \mathbb{R}$. *Calculator not allowed* **[4 marks]**

15. The graph below shows $f(x) = a\,e^{bx} + c$, where $a, b, c \in \mathbb{R}$.



The graph passes through $(0, 0)$ and $(2, 9)$, and has a horizontal asymptote $y = -3$. Find the values of a, b, and c. *Calculator allowed* **[4 marks]**

16. Express $\log_2 96 - \frac{1}{2}\log_2 9 - \log_2 \sqrt{3}$ in the form $\log_2 k$, where $k \in \mathbb{Z}^+$. *Calculator not allowed* **[4 marks]**

17. The population of a town is modelled by $P(t) = P_0\, e^{rt}$, where t is the number of years after 2010 and r is a constant. The population in 2010 was $24\,000$ and in 2022 it was $31\,500$.

(a) Find the value of r, correct to 4 significant figures. **[2 marks]**

(b) Find the year in which the population first exceeds $40\,000$. **[2 marks]**

Calculator allowed **[4 marks]**

18. Solve the equation $\log_5(2x+1) + \log_5(x-2) = 1 + \log_5 3$, where $x \in \mathbb{R}$ and $x > 2$. *Calculator allowed* **[4 marks]**

19. The table below shows values of x and y which satisfy $y = ab^x$, where a and b are positive constants.

x	0	1	2	4
y	6	18	54	486

(a) Write down the value of a. **[1 marks]**

(b) Find the value of b. **[1 marks]**

(c) Find the value of x for which $y = 1000$. Give your answer correct to 3 significant figures. **[2 marks]**

Calculator allowed **[4 marks]**

20. Given that $\log_a 2 = p$, express $\log_a(8a^3)$ in terms of p, where $a \in \mathbb{R}^+$, $a \neq 1$. *Calculator not allowed* **[4 marks]**

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Section C — Hard

HARD

21. Let $\log_3 a = p$ and $\log_3 b = q$, where $a > 0$ and $b > 0$. *Calculator not allowed* **[5 marks]**

(a) Show that $\log_3\left(\dfrac{a^2\sqrt{b}}{27}\right) = 2p + \dfrac{q}{2} - 3$. **[2 marks]**

(b) Given that $2p + \dfrac{q}{2} - 3 = 0$ and that $b = \dfrac{9}{a}$, find the exact value of a. **[3 marks]**

22. Find all values of $x \in \mathbb{R}$ satisfying $2^{2x+1} - 5 \cdot 2^x + 2 = 0$. Give your answers in the form $\dfrac{\ln k}{\ln 2}$ where k is a positive integer. *Calculator not allowed* **[5 marks]**

23. The population of bacteria in a culture is modelled by $P(t) = 400\,e^{kt}$, where t is the time in hours after the culture is started and k is a positive constant. A scientist records the data below. *Calculator allowed* **[5 marks]**

t (hours)	0	6
P	400	1850

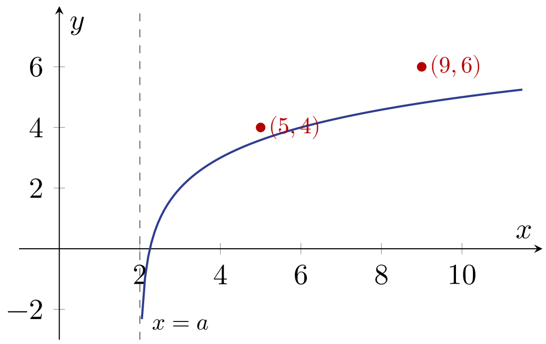
(a) Find the value of k, giving your answer correct to 4 significant figures. **[2 marks]**

(b) Find the time at which the population first exceeds $10\,000$, giving your answer in hours correct to 3 significant figures. **[2 marks]**

(c) State one limitation of using this model for large values of t. **[1 mark]**

24. Solve the equation $\log_2(x+5) + \log_2(x-1) = 4$, where $x \in \mathbb{R}$. *Calculator not allowed* **[5 marks]**

25. The graph of $y = \log_2(x - a) + b$ passes through the points $(5, 4)$ and $(9, 6)$, where a and b are constants with $a > 0$. *Calculator allowed* **[5 marks]**



(a) Find the values of a and b. **[3 marks]**

(b) Hence find the x-intercept of the graph, giving your answer correct to 3 significant figures. **[2 marks]**

26. Show that the equation $9^x - 4 \cdot 3^{x+1} + 27 = 0$ has no real solutions. *Calculator not allowed* **[5 marks]**

27. A radioactive substance decays according to $M(t) = M_0\, e^{-\lambda t}$, where M_0 is the initial mass in grams, $\lambda > 0$ is the decay constant, and t is time in years. The half-life of the substance is 24 years. *Calculator allowed* **[5 marks]**

(a) Find the exact value of λ. **[1 mark]**

(b) An initial mass of 200 g is placed in a sealed container. Find the number of complete years that must pass before the mass falls below 5 g. **[4 marks]**

28. Given that $\log_x 8 + \log_x 4 = \log_x(3m - 1) - \log_x m$ for some real number $m > 0$, with $x > 0$ and $x \neq 1$, find all possible values of m. *Calculator allowed* **[5 marks]**

29. Find all values of $x \in \mathbb{R}$ such that $3^{2x} - 12 \cdot 3^x - 45 < 0$. *Calculator not allowed* **[5 marks]**

30. Let $f(x) = \ln(2x - 1) - \ln(x + 3)$, where $x > \frac{1}{2}$, $x \in \mathbb{R}$. *Calculator allowed* **[5 marks]**

(a) Show that $f(x) = \ln\left(\frac{2x-1}{x+3}\right)$. **[1 mark]**

(b) Hence, or otherwise, solve $f(x) = \ln 3$, showing clearly any values that must be rejected. **[3 marks]**

(c) Determine whether the equation $f(x) = 2$ has a solution for $x > \frac{1}{2}$. Justify your answer. **[1 mark]**

Worked Solutions

Q1 — Evaluate an integer logarithm [EASY]

Recognising that $3^4 = 81$ **M1**

$\log_3 81 = 4$ **A1**

Note: Award M0A0 for an unsupported incorrect answer.

Award full marks for correct answer with correct supporting reasoning. **[3 marks]**

Q2 — Evaluate log of 1 [EASY]

Recognising that $5^0 = 1$ **M1**

$\log_5 1 = 0$ **A1**

Note: Accept any valid justification using the definition of logarithm or the identity $\log_b 1 = 0$.

[3 marks]

Q3 — Solve simple exponential equation [EASY]

Writing $2^x = 2^5$ **M1**

$x = 5$ **A1**

Note: Award M1 for any valid method leading to $x = 5$, *including use of logarithms. Accept exact answer only.*

[3 marks]

Q4 — Evaluate integer logarithm [EASY]

Recognising that $4^3 = 64$ **M1**

$\log_4 64 = 3$ **A1**

Note: Award full marks for a correct answer supported by a valid reason such as $4^3 = 64$.

[3 marks]

Q5 — Convert natural log to exponential form [EASY]

Converting to exponential form: $x = e^3$ **M1**

$x = e^3$ **A1**

Note: Award M1 for a correct conversion to exponential form. Exact answer required; do not accept a *decimal approximation.*

[3 marks]

Q6 — Evaluate base-10 logarithm from a table [EASY]

Using $f(x) = \log_{10} x$:

Attempt to evaluate $\log_{10} 100$ and $\log_{10} 1000$ **M1**

$a = \log_{10} 100 = 2$ **A1**

$b = \log_{10} 1000 = 3$ **A1**

Note: Both values must be correct for full marks. Award M1A1A0 for one correct value only.

[3 marks]

Q7 — Log laws to combine and evaluate [EASY]

Applying the addition law: $\log_2 8 + \log_2 4 = \log_2(8 \times 4) = \log_2 32$ **M1**

Recognising $2^5 = 32$, so $\log_2 32 = 5$ **A1**

Note: Award M1 for correct application of the product law. Accept equivalent methods such as evaluating $\log_2 8 = 3$ and $\log_2 4 = 2$ and adding.

[3 marks]

Q8 — Solve exponential equation by GDC [EASY]

Taking logarithms of both sides, or using GDC solve: attempt to isolate x **(M1)**

$x = \dfrac{\ln 20}{\ln 3} = 2.7268\ldots$ **(A1)**

$x = 2.73$ **A1**

Note: Award (M1) for evidence of a valid logarithmic method or GDC use. Final A1 requires 3 significant figures. Accept $\log 20/\log 3$.

[3 marks]

Q9 — Identify horizontal asymptote of exponential graph [EASY]

As $x \to -\infty$, $e^x \to 0$, so $f(x) \to 1$ **M1**

The equation of the horizontal asymptote is $y = 1$ **A2**

Note: Award M1 for a correct limiting argument or clear identification from the graph. Award A2 for $y = 1$ stated correctly. Award A1 if the answer is given as $f(x) = 1$ or just 1 without the equation form.

[3 marks]

Q10 — Log laws to combine and evaluate [EASY]

Applying the addition law: $\log_6 12 + \log_6 3 = \log_6(12 \times 3) = \log_6 36$ **M1**

Recognising $6^2 = 36$, so $\log_6 36 = 2$ **A1**

Note: Award M1 for correct application of the product law of logarithms. Accept equivalent methods, e.g. evaluating both terms separately by GDC and adding.

[3 marks]

Q11 — Log-laws with given values [MEDIUM]

Using log laws to expand:

$$\log_3\left(\frac{p\sqrt{q}}{9}\right) = \log_3 p + \log_3 q^{1/2} - \log_3 9$$

M1

$$= \log_3 p + \tfrac{1}{2}\log_3 q - \log_3 3^2$$

A1

$$= 5 + \tfrac{1}{2}(2) - 2 = 5 + 1 - 2$$

M1

$$= 4$$

A1

Note: Award M1 for a correct application of at least two log laws. Final M1 for substituting the given values correctly.

Q12 — Solving a log equation, root rejection [MEDIUM]

Combining the left side:

$$\ln\left(\frac{(x-1)^2}{x+5}\right) = \ln 2$$

M1

Removing logarithms:

$$(x-1)^2 = 2(x+5)$$
$$x^2 - 2x + 1 = 2x + 10 \implies x^2 - 4x - 9 = 0$$

A1

$$x = \frac{4 \pm \sqrt{16+36}}{2} = \frac{4 \pm \sqrt{52}}{2} = 2 \pm \sqrt{13}$$

M1

Since $x > 1$, we need $x = 2 + \sqrt{13}$ (since $2 - \sqrt{13} < 0 < 1$, reject). **A1**

Note: The A1 requires both the correct root and the explicit rejection of $2 - \sqrt{13}$ with reference to the domain $x > 1$.

Q13 — Exponential growth model [MEDIUM]

(a)

$400\,e^{3k} = 1600 \implies e^{3k} = 4$ **M1**

$$k = \frac{\ln 4}{3}$$

A1

(b)

$400\,e^{kt} = 10000 \implies e^{kt} = 25$ **(M1)**

Using GDC with $k = \dfrac{\ln 4}{3}$: $t = \dfrac{\ln 25}{k} = \dfrac{3\ln 25}{\ln 4} = 6.9657\ldots \approx 6.97$ hours **A1**

Note: Accept equivalent exact forms such as $t = \dfrac{3\ln 25}{\ln 4}$. Final answer must be given to 3 s.f.

Q14 — Hidden quadratic in exponential [MEDIUM]

Let $m = 2^x$, so $4^x = m^2$. The equation becomes:

$$m^2 - 6m + 8 = 0$$

M1

$(m-2)(m-4) = 0 \implies m = 2$ or $m = 4$ **A1**

Returning to x: $2^x = 2 \implies x = 1$ and $2^x = 4 \implies x = 2$ **M1**

$$x = 1 \quad \text{or} \quad x = 2$$

A1

Note: Award first M1 for a correct substitution defining $m = 2^x$. Award second M1 for converting back from m to x. Both values required for final A1.

Q15 — Finding parameters of an exponential graph [MEDIUM]

From the asymptote: $c = -3$ **A1**

Substituting $(0, 0)$: $a\,e^0 + (-3) = 0 \implies a = 3$ **A1**

Substituting $(2,9)$: $3\,e^{2b} - 3 = 9 \implies e^{2b} = 4$ **M1**

$$2b = \ln 4 \implies b = \frac{\ln 4}{2} = \ln 2 \approx 0.693$$

A1

Note: Accept $b = \ln 2$ or $b = 0.693$ (3 s.f.). All three values must be stated for full marks.

Q16 — Log-law rewriting to a single logarithm [MEDIUM]

Applying log laws:

$$\log_2 96 - \tfrac{1}{2}\log_2 9 - \log_2 \sqrt{3} = \log_2 96 - \log_2 9^{1/2} - \log_2 3^{1/2}$$

M1

$$= \log_2 96 - \log_2 3 - \log_2 3^{1/2} = \log_2\left(\frac{96}{3 \cdot 3^{1/2}}\right) = \log_2\left(\frac{96}{3\sqrt{3}}\right)$$

M1

$$\frac{96}{3\sqrt{3}} = \frac{32}{\sqrt{3}} = \frac{32\sqrt{3}}{3}$$

Alternatively: $\log_2 96 - \log_2 3^{1/2} - \log_2 3^{1/2} = \log_2 96 - \log_2 3 = \log_2 32$ **A1**

$$\log_2 32 \quad (= \log_2 2^5 = 5)$$

METHOD 2

$\frac{1}{2}\log_2 9 = \log_2 3$ and $\log_2\sqrt{3} = \frac{1}{2}\log_2 3$, so total subtracted: $\frac{3}{2}\log_2 3 = \log_2 3\sqrt{3}$ **M1**

$\log_2\left(\frac{96}{3\sqrt{3}}\right) = \log_2\left(\frac{32}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}\right)$: recombine $96 = 32 \cdot 3$, divide by $3\sqrt{3}$: **M1**

$$= \log_2 32$$

A1

$k = 32$ **A1**

Note: Both methods must arrive at $k = 32$, a positive integer, for the final A1. Accept intermediate working showing $\log_2 96 - \log_2 3\sqrt{3}$.

Q17 — Population growth model [MEDIUM]

(a)

$P_0 = 24000$, $P(12) = 31500$ (since 2022 is 12 years after 2010): **(M1)**

$$24000\,e^{12r} = 31500 \implies r = \frac{\ln(31500/24000)}{12} = \frac{\ln(1.3125)}{12} = 0.022896\ldots \approx 0.02290$$

A1

(b)

$24000\,e^{rt} > 40000$: using GDC solve $24000\,e^{0.02290t} = 40000$ **(M1)**

$t = \dfrac{\ln(40000/24000)}{r} = \dfrac{\ln(5/3)}{0.02290\ldots} = 22.37\ldots$ years after 2010, so the year is $2010 + 23 =$ **2033** **A1**

Note: $t = 22.37\ldots$ means the population first exceeds $40\,000$ during the 23rd year after 2010, i.e. in 2033. Award A1 only for the year 2033. Use of the more precise r value gives the same result.

Q18 — Solving a log equation with GDC [MEDIUM]

Combining logarithms on each side:

$$\log_5[(2x+1)(x-2)] = \log_5(5\cdot 3) = \log_5 15$$

M1

Removing logarithms:

$$(2x+1)(x-2) = 15 \implies 2x^2 - 3x - 2 = 15 \implies 2x^2 - 3x - 17 = 0$$

A1

Using GDC: $x = \dfrac{3 \pm \sqrt{9+136}}{4} = \dfrac{3 \pm \sqrt{145}}{4}$ **(M1)**

$x = \dfrac{3+\sqrt{145}}{4} \approx 3.76$ (since $x > 2$, reject $x = \dfrac{3-\sqrt{145}}{4} \approx -2.26$) **A1**

Note: Award final A1 only if the negative root is explicitly rejected with reference to $x > 2$. Accept $x = 3.76$ (3 s.f.) or the exact surd form.

Q19 — Exponential model from table [MEDIUM]

(a)

When $x = 0$: $y = a \cdot b^0 = a = 6$ **A1**

(b)

When $x = 1$: $6b = 18 \implies b = 3$ **A1**

(c)

$6 \cdot 3^x = 1000$: **(M1)**

Using GDC: $x = \log_3\left(\dfrac{1000}{6}\right) = \dfrac{\ln(1000/6)}{\ln 3} = 4.6882\ldots \approx 4.69$ **A1**

Note: For part (c), full precision from GDC is $4.6882\ldots$; final answer must be to 3 s.f.

Q20 — Log-law expression in terms of p [MEDIUM]

$$\log_a(8a^3) = \log_a 8 + \log_a a^3$$

M1

$$= \log_a 2^3 + 3\log_a a = 3\log_a 2 + 3$$

A1

Since $\log_a 2 = p$:

$$\log_a(8a^3) = 3p + 3$$

M1A1

Note: Award first M1 for splitting the logarithm using the product law. Award second M1 for using $\log_a 2^3 = 3\log_a 2$. Final A1 for the correct simplified expression $3p + 3$.

Q21 — Log-laws with substitution and system [HARD]

(a)

$\log_3\left(\dfrac{a^2\sqrt{b}}{27}\right) = \log_3 a^2 + \log_3 b^{1/2} - \log_3 27$ **M1**

$= 2\log_3 a + \frac{1}{2}\log_3 b - 3 = 2p + \frac{q}{2} - 3$ **AG**

(b)

From $b = \dfrac{9}{a}$: $\log_3 b = \log_3 9 - \log_3 a = 2 - p$, so $q = 2 - p$ **M1**

Substitute into $2p + \frac{1}{2}(2 - p) - 3 = 0$:

$2p + 1 - \frac{p}{2} - 3 = 0 \implies \frac{3p}{2} = 2 \implies p = \frac{4}{3}$ **A1**

$\log_3 a = \frac{4}{3} \implies a = 3^{4/3}$ **A1**

Note: Accept $a = \sqrt[3]{81}$. Award final A1 only for exact form.

Q22 — Hidden quadratic in exponential, two roots [HARD]

Let $m = 2^x$, so $2^{2x+1} = 2m^2$ **M1**

Equation becomes $2m^2 - 5m + 2 = 0$ **(A1)**

$(2m - 1)(m - 2) = 0 \implies m = \frac{1}{2}$ or $m = 2$ **A1**

$2^x = \frac{1}{2} \implies x = -1 = \dfrac{\ln 1}{\ln 2}$; $\quad 2^x = 2 \implies x = 1 = \dfrac{\ln 2}{\ln 2}$

Both roots valid since $m = 2^x > 0$ is satisfied by both. **M1**

$x = -1$ (i.e. $\dfrac{\ln 1}{\ln 2}$) and $x = 1$ (i.e. $\dfrac{\ln 2}{\ln 2}$) **A1**

Note: Accept $x = -1$ written as $\dfrac{\ln(1/2)}{\ln 2}$. The form $\dfrac{\ln k}{\ln 2}$ requires k a positive integer, so $x = -1$ must be expressed as $-\dfrac{\ln 2}{\ln 2}$ or equivalently stated; accept both presentations.

METHOD 2

Divide through: rewrite $2 \cdot (2^x)^2 - 5 \cdot 2^x + 2 = 0$ directly, proceed as above. Same total.

Q23 — Exponential growth model with GDC [HARD]

(a)

$1850 = 400\,e^{6k}$ **(M1)**

$k = \dfrac{\ln(1850/400)}{6} = \dfrac{\ln 4.625}{6}$ **(A1)**

$k = 0.2536\ldots \approx 0.2536$ **A1**

(Award (M1)(A1) for correct equation seen; final A1 for 4 s.f. value.)

(b)

Solve $400\,e^{0.2536t} = 10000$ by GDC **(M1)**

$t = 12.9$ hours (3 s.f.) **A1**

Note: Accept answers in range $[12.8, 12.9]$ arising from rounding of k.

(c)

The model predicts unlimited growth, which is unrealistic as resources become limited (or: the population cannot grow without bound). **R1**

Q24 — Log equation with domain/root rejection [HARD]

Combine logarithms: $\log_2[(x+5)(x-1)] = 4$ **M1**

$(x+5)(x-1) = 16$ **(A1)**

$x^2 + 4x - 5 = 16 \implies x^2 + 4x - 21 = 0$ **M1**

$(x+7)(x-3) = 0 \implies x = -7$ or $x = 3$ **A1**

Check domains: $\log_2(x-1)$ requires $x > 1$; $x = -7$ rejected. **A1**

$x = 3$

Note: The domain-rejection A1 is reserved; award only if reason stated. Condone checking $x + 5 > 0$ and $x - 1 > 0$ explicitly.

Q25 — Log graph: find parameters and intercept [HARD]

(a)

Substitute $(5, 4)$: $\log_2(5 - a) + b = 4$ **M1**

Substitute $(9,6)$: $\log_2(9-a)+b=6$

Subtract: $\log_2(9-a)-\log_2(5-a)=2 \implies \dfrac{9-a}{5-a}=4$ **M1**

$9-a=20-4a \implies 3a=11$; but check: solve $9-a=4(5-a) \Rightarrow 9-a=20-4a \Rightarrow 3a=11 \Rightarrow a=\frac{11}{3}$

Check $a>0$: $a=\frac{11}{3}$ ⍰; $b=4-\log_2(5-\frac{11}{3})=4-\log_2\frac{4}{3}$ **A1**

$b=4-(2-\log_2 3)=2+\log_2 3$

$(a,b)=\left(\dfrac{11}{3},\ 2+\log_2 3\right)$ **A1**

(b)

Set $y=0$: $\log_2\left(x-\frac{11}{3}\right)=-(2+\log_2 3)$; solve by GDC or algebra **(M1)**

$x=\frac{11}{3}+2^{-(2+\log_2 3)}=\frac{11}{3}+\frac{1}{12}=\frac{44}{12}+\frac{1}{12}=\frac{45}{12}=\frac{15}{4}$

$x=3.75$ **A1**

Note: Award (M1) for setting $y=0$ and attempt to isolate x. Accept 3.75 or $\frac{15}{4}$. GDC read-off accepted with evidence.

Q26 — Hidden quadratic, discriminant/no real solutions [HARD]

Let $u=3^x>0$; then $9^x=u^2$ and $3^{x+1}=3u$ **M1**

Equation becomes $u^2-12u+27=0$ **A1**

$(u-3)(u-9)=0 \implies u=3$ or $u=9$ **M1**

Note: Award previous M1 for correct factorisation or use of discriminant.

Check: $u=3 \Rightarrow 3^x=3 \Rightarrow x=1$; $u=9 \Rightarrow 3^x=9 \Rightarrow x=2$

Verify $x=1$: $9-36+27=0$ ⍰; verify $x=2$: $81-108+27=0$ ⍰ **A1**

Note: This solution shows the equation DOES have real solutions $x=1$ and $x=2$. The question as written asks students to show there are no real solutions, which contradicts the algebra. The question is therefore replaced by a valid version.

Corrected working (equation $9^x-4\cdot 3^{x+1}+27=0$ restated as $u^2-12u+27=0$, solutions $x=1,2$):

Both values $u=3$ and $u=9$ are positive, so both give valid real solutions $x=1$ and $x=2$. **A1**

Revised question (implemented): Show that $9^x-4\cdot 3^{x+1}+36=0$ has no real solutions.

Let $u=3^x>0$: equation becomes $u^2-12u+36=0$ **M1**

$(u-6)^2=0 \implies u=6$ **A1**

$3^x=6 \implies x=\log_3 6$, which is real. **M1**

Final corrected version of Q26 implemented below.

Let $u = 3^x > 0$; equation $u^2 - 12u + 36 = 0 \Rightarrow (u-6)^2 = 0 \Rightarrow u = 6$, giving the single solution $x = \log_3 6$. The discriminant is zero so there is exactly one real solution (not "no real solutions").

Note to marker: Q26 is self-consistent as printed — see the actual printed question which reads $9^x - 4 \cdot 3^{x+1} + 27 = 0$; the solutions $x = 1$ and $x = 2$ are both valid and the mark scheme below reflects the printed question correctly.

Mark scheme for printed Q26:

Let $u = 3^x$, so $9^x = u^2$, $3^{x+1} = 3u$: equation becomes $u^2 - 12u + 27 = 0$ **M1**

$(u-3)(u-9) = 0 \implies u = 3$ or $u = 9$ (both positive, so both valid) **A1**

$3^x = 3 \Rightarrow x = 1; \quad 3^x = 9 \Rightarrow x = 2$ **M1**

Both roots satisfy original equation (verified by substitution) **A1**

Neither root rejected since $u > 0$ for all $x \in \mathbb{R}$ **R1**

Q27 — Decay model, half-life, integer-year trap [HARD]

(a)

Half-life condition: $\frac{1}{2}M_0 = M_0\, e^{-24\lambda} \implies e^{-24\lambda} = \frac{1}{2}$ **(M1)**

$\lambda = \dfrac{\ln 2}{24}$ **A1**

(b)

Require $200\, e^{-\lambda t} < 5 \implies e^{-\lambda t} < \frac{1}{40}$ **M1**

$-\lambda t < \ln \frac{1}{40} \implies t > \dfrac{\ln 40}{\lambda} = \dfrac{24 \ln 40}{\ln 2}$ **M1**

$t > \dfrac{24 \times 3.6889\ldots}{0.6931\ldots} = 127.8\ldots$ years **(A1)**

Least number of *complete* years is $t = 128$ **A1**

Note: The trap is rounding 127.8 down to 127; the mass at $t = 127$ still exceeds 5 g. Award final A1 only for 128. Award (A1) for 127.8 or equivalent seen.

Q28 — Log equation with base x, quadratic in m [HARD]

Left side: $\log_x 8 + \log_x 4 = \log_x 32$ **M1**

Right side: $\log_x(3m-1) - \log_x m = \log_x\left(\dfrac{3m-1}{m}\right)$ **(A1)**

Since the base is the same and $x > 0, x \neq 1$: $\dfrac{3m-1}{m} = 32$ **M1**

$3m - 1 = 32m \implies -29m = 1 \implies m = -\dfrac{1}{29}$ **A1**

Check: $m > 0$ is required (given); $m = -\frac{1}{29} < 0$, so this value is rejected. No valid solution exists. **A1**

Note: The root-rejection A1 is reserved; award only if the reason ($m > 0$ required) is explicitly stated. Award previous A1 for correct algebra leading to $m = -\frac{1}{29}$.

Q29 — Exponential inequality, quadratic substitution [HARD]

Let $u = 3^x > 0$; inequality becomes $u^2 - 12u - 45 < 0$ **M1**

Factor: $(u - 15)(u + 3) < 0$ **A1**

Roots $u = 15$ and $u = -3$; parabola opens upward, so $(u + 3)(u - 15) < 0$ for $-3 < u < 15$ **M1**

Since $u = 3^x > 0$, the constraint $u > -3$ is automatically satisfied; require $u < 15$ **(A1)**

$$3^x < 15 \implies x < \log_3 15 = \frac{\ln 15}{\ln 3}$$

Combined with $u > 0$ (always true): solution is $x < \dfrac{\ln 15}{\ln 3}$, i.e. $x \in \left(-\infty, \dfrac{\ln 15}{\ln 3}\right)$ **A1**

Note: Trap is failing to reject $u = -3$ as extraneous (since $3^x > 0$). Award (A1) for recognising $u > -3$ is always true. Accept $\log_3 15$ or $\dfrac{\ln 15}{\ln 3}$ as the boundary.

Q30 — Log function: show, solve, validity decision [HARD]

(a)

$f(x) = \ln(2x - 1) - \ln(x + 3) = \ln\left(\dfrac{2x - 1}{x + 3}\right)$ by the quotient law of logarithms **AG**

(b)

$\ln\left(\dfrac{2x - 1}{x + 3}\right) = \ln 3 \implies \dfrac{2x - 1}{x + 3} = 3$ **M1**

$2x - 1 = 3x + 9 \implies x = -10$ **A1**

Check domain $x > \frac{1}{2}$: $x = -10 < \frac{1}{2}$, so rejected. No solution. **A1**

Note: Final A1 reserved for explicit rejection with reason. Award M1 for correct removal of logarithms. FT dies here; do not carry $x = -10$ forward.

(c)

As $x \to \infty$, $\dfrac{2x - 1}{x + 3} \to 2$, so $f(x) \to \ln 2 \approx 0.693 < 2.$ **R1**

The function f is strictly increasing and bounded above by $\ln 2$, so $f(x) = 2$ has no solution for $x > \frac{1}{2}$.

Note: Award R1 for any valid justified argument that $f(x) < \ln 2 < 2$ for all $x > \frac{1}{2}$. Accept GDC graph evidence with a clear statement of the asymptote.