

IB Math AA SL — Topic 1 Proof & Reasoning

Direct Algebraic Proof · Identities · Counterexamples

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name: _____ Date: _____
Score: _____/120



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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.



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Key Formulae

- Even integer: $n = 2k, k \in \mathbb{Z}$; Odd integer: $n = 2k + 1, k \in \mathbb{Z}$
- Consecutive integers: $n, n + 1, n + 2, \dots$, where $n \in \mathbb{Z}$
- Difference of two squares: $a^2 - b^2 = (a - b)(a + b)$
- Completing the square: $ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}$; expression is always positive iff $a > 0$ and discriminant $b^2 - 4ac < 0$
- Divisibility form: to show E is divisible by d , write $E = d \cdot (\text{integer expression})$
- Disproof by counterexample: find one specific n for which $P(n)$ is false and compute both sides explicitly to exhibit the contradiction
- Sum/product parity rules: odd + odd = even; odd $\times$ odd = odd; even $\times$ anything = even

GDC Tips

- **Integer table exploration:** use the Table function ($\text{TblStart} = 1, \Delta\text{Tbl} = 1$) to evaluate an expression for $n = 1, 2, \dots, 10$ before conjecturing a divisibility pattern.
- **Counterexample search:** store a candidate expression as Y_1 and scroll the table to locate the first n for which the claim fails; record the numerical value as your counterexample.
- **Verify a specific case:** use the home-screen $\text{STO} \rightarrow$ key to assign $n = k$ and evaluate both sides of an identity to check a single substitution (verify, not prove).
- **Remainder check:** compute $\text{remainder}(\text{expr}, d)$ or use $\text{iPart} / \text{mod}(\text{expr}, d)$ to test divisibility for several values during exploration.
- **Expand & simplify on CAS:** use $\text{expand}()$ and $\text{factor}()$ on a CAS-enabled GDC to confirm algebraic manipulations; copy the exact form into written working.

Common Mistakes to Avoid

1. **Failing to define the integer variable.** Writing “let n be even” without stating $n = 2k$ for some $k \in \mathbb{Z}$ before any manipulation loses the opening mark; the proof has no declared object.
2. **Confusing verification with proof.** Checking that a statement holds for $n = 1, 2, 3$ is not a general proof; a “show that” or “prove” command requires an algebraic argument valid for all integers, not a numerical check.
3. **Assuming the target in a “show that” question.** Beginning with the printed result and working backwards (or writing both sides and equating them) is circular reasoning; always start from one side and reach the other by a forward chain.
4. **Incomplete counterexample.** Stating only the value of n without computing both sides of the claimed inequality or identity and explicitly identifying the contradiction does not earn the mark; the contradiction must be stated in words.
5. **Omitting the conclusion line.** Every proof must end with a sentence restating the claim (e.g. “therefore $n^2 - n$ is always even”); stopping at the factored expression without the conclusion loses the final reasoning mark R1.
6. **Wrong completing-the-square sign decision.** When showing an expression is always positive, students write $a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}$ but then claim the constant term is positive without checking $b^2 < 4ac$; the sign of the discriminant must be verified explicitly to justify the conclusion.

Section A — Easy **EASY** (10 questions $\times$ 3 marks = 30 marks)



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1. **EASY** No Calculator [3 marks]

Let n be an integer. Show that the sum of n and $n + 2$ is always even.

2. **EASY** No Calculator [3 marks]

Let n be an integer. Show that the product of two consecutive integers, n and $n + 1$, is always even.

3. **EASY** Calculator [3 marks]

Consider the statement: "For all integers n , the expression $n^2 + n + 1$ is prime." Disprove this statement by finding a counterexample, where $n \in \mathbb{Z}^+$.

4. **EASY** No Calculator [3 marks]

Let n be an integer. Show that $n^2 - n$ is always even.

5. **EASY** **Calculator**

[3 marks]

Determine whether the following statement is true or false: "The sum of any three consecutive integers is divisible by 3." Justify your answer.

6. **EASY** **Calculator**

[3 marks]

The following table shows a numerical exploration of the expression $n^2 - 1$ for positive integer values of n , where $n \geq 2$.

n	2	3	4	5
$n^2 - 1$	3	8	15	24
$(n - 1)(n + 1)$	3	8	15	24

Using the table as inspiration, show that $n^2 - 1 = (n - 1)(n + 1)$ for all integers n .

7. **EASY** **Calculator**

[3 marks]

Consider the statement: "For all real numbers x , $x^2 \geq x$." Disprove this statement by finding a counterexample, where $x \in \mathbb{R}$.

8. **EASY** **No Calculator**

[3 marks]

Let n be an integer. Show that the product of two odd integers, $2n + 1$ and $2n + 3$, is always odd.

9. **EASY** **Calculator**

[3 marks]

Consider the expression $x^2 + 6x + 13$, where $x \in \mathbb{R}$. By completing the square, show that $x^2 + 6x + 13 > 0$ for all real values of x .

10. **EASY** **Calculator**

[3 marks]

Verify that $n = 4$ satisfies the equation $n^2 - 5n + 4 = 0$.

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)



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11. **MEDIUM** No Calculator [4 marks]

Let n be an integer. Show that the product of any two consecutive even integers is divisible by 8.

12. **MEDIUM** Calculator [4 marks]

Let n be an integer. Prove that $n^2 + n$ is always even.

13. **MEDIUM** Calculator [4 marks]

Consider the statement: "For all integers n , if n^2 is divisible by 4, then n is divisible by 4." Disprove this statement by finding a counterexample. Justify your answer.

14. **MEDIUM** No Calculator

[4 marks]

Let n be an integer. Prove that the difference of the squares of any two consecutive odd integers is divisible by 8.

15. **MEDIUM** Calculator

[4 marks]

The table below shows the values of n and $f(n) = n^2 - n + 11$ for $n \in \{1, 2, 3, 4, 5\}$.

n	1	2	3	4	5
$f(n)$	11	13	17	23	31

A student conjectures that $f(n) = n^2 - n + 11$ is prime for all positive integers n .

(a) Verify the conjecture for $n = 5$.

[1]

(b) Disprove the conjecture by finding a value of n for which $f(n)$ is not prime.

[3]

16. **MEDIUM** No Calculator

[4 marks]

Let a and b be real numbers where $a > 0$ and $b > 0$. Show that $\frac{a}{b} + \frac{b}{a} \geq 2$.

17. MEDIUM Calculator

[4 marks]

Let n be an integer. Prove that $(n + 3)^2 - (n - 1)^2$ is always divisible by 8.

18. MEDIUM No Calculator

[4 marks]

Consider the expression $f(x) = 3x^2 - 12x + 17$, where $x \in \mathbb{R}$. Show that $f(x) > 0$ for all real values of x .

19. MEDIUM Calculator

[4 marks]

Let n be a positive integer. Consider the statement: " $4^n - 1$ is divisible by 3 for all positive integers n ."

(a) Verify the statement for $n = 3$.

[1]

(b) Prove the statement by writing $4^n - 1$ in a suitable factored or expanded form.

[3]

20. MEDIUM Calculator

[4 marks]

Let m and n be integers. Prove that if m is odd and n is odd, then $m^2 + n^2 + 2$ is divisible by 4.

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)



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21. **HARD** No Calculator [5 marks]

Let $n \in \mathbb{Z}^+$. Show that $n^3 - n$ is divisible by 6 for all positive integers n .

22. **HARD** No Calculator [5 marks]

Let p and q be odd integers. Prove that $p^2 + q^2 + 2$ is divisible by 4.

23. **HARD** Calculator

[5 marks]

Consider the statement: “For all integers n where $n \geq 2$, the expression $n^2 - n + 41$ is prime.” Determine, with full justification, whether this statement is true or false.

24. **HARD** No Calculator

[5 marks]

Let $a, b \in \mathbb{R}$ with $a \neq b$. Show that $a^2 + ab + b^2 > 0$.

25. **HARD** Calculator

[5 marks]

Aiko claims that for all real numbers x and y , $(x + y)^2 \geq 4xy$. Determine whether Aiko's claim is true or false. If true, prove the result. If false, provide a counterexample and state the correct condition on x and y under which equality holds.

26. **HARD** Calculator

[5 marks]

The following table shows values of $f(n) = 2^n + 6$ for selected positive integers n .

n	1	2	3	4
$f(n)$	8	10	14	22

A student conjectures that $f(n) = 2^n + 6$ is always even for $n \in \mathbb{Z}^+$. Determine whether this conjecture is true, and prove your conclusion.

27. **HARD** No Calculator

[5 marks]

Let n be an integer with $n \geq 1$. Show that $(2n + 1)^2 - (2n - 1)^2 = 8n$, and hence prove that the sum of the first N positive odd integers is N^2 , where $N \in \mathbb{Z}^+$.

28. **HARD** No Calculator

[5 marks]

Let a and b be positive real numbers. Prove that $\frac{a}{b} + \frac{b}{a} \geq 2$, stating clearly when equality holds.

29. **HARD** Calculator

[5 marks]

Consider the quadratic expression $f(x) = kx^2 - 4x + k$ where $k \in \mathbb{R}$, $k \neq 0$. Find the values of k for which $f(x) > 0$ for all $x \in \mathbb{R}$, justifying your answer fully.

30. **HARD** No Calculator

[5 marks]

Prove that for any two distinct real numbers a and b , the arithmetic mean of a and b is strictly greater than their geometric mean when both a and b are positive and $a \neq b$. That is, prove that $\frac{a+b}{2} > \sqrt{ab}$ for all $a, b \in \mathbb{R}^+$ with $a \neq b$.

Mark Schemes

Q1 — Sum of Two Integers Two Apart is Even EASY

Let n be an integer, so the two integers are n and $n + 2$. **M1**
Their sum is $n + (n + 2) = 2n + 2 = 2(n + 1)$. **A1**
Since $2(n + 1)$ is a multiple of 2, the sum is always even. **R1**
Note: The conclusion must be stated in words or by reference to the factor of 2; accept equivalent phrasing.

Q2 — Product of Consecutive Integers is Even EASY

Let n be an integer, so the consecutive integers are n and $n + 1$. **M1**
Either n or $n + 1$ must be even (consecutive integers have opposite parity), so their product $n(n + 1)$ contains a factor of 2. **A1**
Hence $n(n + 1)$ is always even. **R1**
METHOD 2
 $n(n + 1) = n^2 + n$. **M1**
Consider cases: if $n = 2k$ then $n(n + 1) = 2k(2k + 1) = 2(k(2k + 1))$, even; if $n = 2k + 1$ then $n(n + 1) = (2k + 1)(2k + 2) = 2(2k + 1)(k + 1)$, even. **A1**
Hence $n(n + 1)$ is always even. **R1**

Q3 — Counterexample to a Primality Claim EASY

Test $n = 3$: **M1**
 $n^2 + n + 1 = 9 + 3 + 1 = 13$, which is prime.
Test $n = 4$: $16 + 4 + 1 = 21 = 3 \times 7$, which is not prime. **A1**
Since $n = 4$ gives a non-prime value, the statement is false. **R1**
Note: Any valid counterexample earns full marks; accept $n = 7$ ($57 = 3 \times 19$), $n = 10$ ($111 = 3 \times 37$), etc. The contradiction must be stated.

Q4 — n Squared Minus n is Always Even EASY

Let n be an integer. Write $n^2 - n = n(n - 1)$. **M1**
The integers $n - 1$ and n are consecutive, so one of them is even, giving $n(n - 1)$ a factor of 2. **A1**
Hence $n^2 - n$ is always even. **R1**

Q5 — Sum of Three Consecutive Integers Divisible by Three EASY

Let the three consecutive integers be n , $n + 1$, and $n + 2$, where n is an integer. **M1**

Their sum is $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$. **A1**

Since $3(n + 1)$ is a multiple of 3, the statement is true and the sum is always divisible by 3. **R1**

Note: The word “true” or an equivalent verdict must appear; award R1 only if a correct general algebraic argument is present.

Q6 — Algebraic Identity n Squared Minus One EASY

Expand the right-hand side: **M1**

$$(n - 1)(n + 1) = n^2 + n - n - 1 \quad \text{A1}$$

$$= n^2 - 1 \quad \text{AG}$$

which is the left-hand side, hence the identity holds for all integers n . **R1**

Note: Final line is AG and carries no marks; R1 is awarded for a statement that the identity has been established in general (not just for the table values).

Q7 — Disproof of x Squared At Least x EASY

Test $x = 0.5$: **M1**

$$x^2 = 0.25 \text{ and } x = 0.5, \text{ so } x^2 = 0.25 < 0.5 = x. \quad \text{A1}$$

Since $x^2 < x$ when $x = 0.5$, the statement is false. **R1**

Note: Any $x \in (0, 1)$ is a valid counterexample; the computed values must be shown and the contradiction stated explicitly.

Q8 — Product of Two Odd Integers is Odd EASY

Let $2n + 1$ and $2n + 3$ be two odd integers, where n is an integer. **M1**

$$(2n + 1)(2n + 3) = 4n^2 + 6n + 2n + 3 = 4n^2 + 8n + 3 = 2(2n^2 + 4n + 1) + 1 \quad \text{A1}$$

Since this is of the form $2k + 1$ where $k = 2n^2 + 4n + 1$ is an integer, the product is always odd. **R1**

Note: Accept any correct factored form that clearly identifies an odd structure; the conclusion must be stated.

Q9 — Completing the Square Shows Positivity EASY

Attempt to complete the square: **M1**

$$x^2 + 6x + 13 = (x + 3)^2 - 9 + 13 = (x + 3)^2 + 4 \quad \text{A1}$$

Since $(x + 3)^2 \geq 0$ for all $x \in \mathbb{R}$, we have $(x + 3)^2 + 4 \geq 4 > 0$, so $x^2 + 6x + 13 > 0$ for all real x . **R1**
Note: The inequality chain and conclusion must appear; do not award R1 for merely stating the completed-square form without justification.

Q10 — Verify $n = 4$ Satisfies a Quadratic **EASY**

Substitute $n = 4$ into $n^2 - 5n + 4$: **M1**

$$(4)^2 - 5(4) + 4 = 16 - 20 + 4 \quad \text{A1}$$

$$= 0 \quad \text{A1}$$

Since the result equals 0, $n = 4$ satisfies the equation.

Note: All three arithmetic steps must be shown; the conclusion that the result equals 0 must be stated.

Q11 — Product of Consecutive Even Integers Divisible by Eight **MEDIUM**

Let n be an integer. Then two consecutive even integers can be written as $2n$ and $2n + 2$. **M1**

Their product is $2n(2n + 2) = 4n(n + 1)$. **A1**

Since n and $n + 1$ are consecutive integers, one of them must be even, so $n(n + 1)$ is divisible by 2. **M1**

Therefore $4n(n + 1) = 4 \times 2k = 8k$ for some integer k , which is divisible by 8. **A1**

Note: Award M1 for defining consecutive even integers algebraically as $2n$ and $2n + 2$ (or $2k$ and $2k + 2$). Accept equivalent notation. The conclusion must state divisibility by 8 explicitly.

Q12 — $n^2 + n$ is Always Even **MEDIUM**

Let n be an integer. Then $n^2 + n = n(n + 1)$. **M1**

Since n and $n + 1$ are consecutive integers, exactly one of them is even. **R1**

Therefore their product $n(n + 1)$ is even, i.e. $n^2 + n$ is always even. **A1**

METHOD 2

Case 1: Let $n = 2k$. Then $n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$, which is even.

Case 2: Let $n = 2k + 1$. Then $n^2 + n = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$, which is even. **M1 A1**

In both cases $n^2 + n$ is even. **R1**

Note: Both methods earn 3 marks total (M1, R1, A1). Award M1 for a valid attempt to factor or use parity cases. Final conclusion required for A1.

Q13 — Counterexample to a Divisibility Implication MEDIUM

Consider $n = 6$. M1
 Then $n^2 = 36 = 4 \times 9$, so n^2 is divisible by 4. A1
 However, $6 \div 4 = 1.5$, so 6 is not divisible by 4. A1
 This is a counterexample, so the statement is false. R1
Note: Accept any even integer n not divisible by 4, e.g. $n = 2$: $n^2 = 4$ is divisible by 4 but 2 is not divisible by 4. Award R1 only if contradiction is explicitly stated. Do not award R1 for a statement without computation.

Q14 — Difference of Squares of Consecutive Odd Integers MEDIUM

Let n be an integer. Two consecutive odd integers can be written as $2n + 1$ and $2n + 3$. M1
 Their difference of squares is $(2n + 3)^2 - (2n + 1)^2$.
 $= (4n^2 + 12n + 9) - (4n^2 + 4n + 1)$ M1
 $= 8n + 8 = 8(n + 1)$ A1
 Since $n + 1$ is an integer, $8(n + 1)$ is divisible by 8. A1
Note: Award first M1 for defining consecutive odd integers algebraically. Award second M1 for expanding both squares. Final A1 requires the factored form $8(\dots)$ and a conclusion.

Q15 — Conjecture About a Quadratic Being Prime MEDIUM

(a) $f(5) = 25 - 5 + 11 = 31$, which is prime. A1
 (b) Consider $n = 11$: M1
 $f(11) = 121 - 11 + 11 = 121 = 11^2$ A1
 Since $121 = 11 \times 11$ is not prime, the conjecture is disproved. A1
Note: For part (b), award M1 for a systematic attempt to find a counterexample (e.g. testing multiples of 11 or values where $n^2 - n + 11$ shares a factor). Accept any valid counterexample with correct computation and explicit statement that the result is not prime. $n = 11$ is the smallest counterexample.

Q16 — Inequality a over b Plus b over a MEDIUM

Consider $\frac{a}{b} + \frac{b}{a} - 2 = \frac{a^2 + b^2 - 2ab}{ab}$. M1
 $= \frac{(a - b)^2}{ab}$ A1
 Since $a > 0$ and $b > 0$, we have $ab > 0$, and $(a - b)^2 \geq 0$ for all real a, b . R1

Therefore $\frac{(a-b)^2}{ab} \geq 0$, which gives $\frac{a}{b} + \frac{b}{a} \geq 2$. **A1**

Note: Award M1 for combining the fractions over a common denominator ab . Award R1 for explicitly stating both $(a-b)^2 \geq 0$ and $ab > 0$. Final A1 requires a conclusion restating the inequality. Do not award A1 if the direction of argument is reversed (assumes the result).

Q17 — Difference of Squares Divisible by Eight **MEDIUM**

$$(n+3)^2 - (n-1)^2 = (n^2 + 6n + 9) - (n^2 - 2n + 1) \quad \textbf{M1}$$

$$= 8n + 8 \quad \textbf{A1}$$

$$= 8(n+1) \quad \textbf{A1}$$

Since $n+1$ is an integer, $8(n+1)$ is divisible by 8. **R1**

METHOD 2

$$\text{Using difference of two squares: } (n+3)^2 - (n-1)^2 = [(n+3) + (n-1)][(n+3) - (n-1)] \quad \textbf{M1}$$

$$= (2n+2)(4) = 8(n+1) \quad \textbf{A1 A1}$$

Since $n+1$ is an integer, this is divisible by 8. **R1**

Note: Award M1 for expanding or factoring correctly. Award R1 only if a conclusion about divisibility by 8 is explicitly stated. Both methods earn 4 marks.

Q18 — Positivity of a Quadratic Expression **MEDIUM**

$$\text{Attempt to complete the square: } 3x^2 - 12x + 17 = 3(x^2 - 4x) + 17 \quad \textbf{M1}$$

$$= 3(x-2)^2 - 12 + 17 \quad \textbf{A1}$$

$$= 3(x-2)^2 + 5 \quad \textbf{A1}$$

Since $(x-2)^2 \geq 0$ for all $x \in \mathbb{R}$, we have $3(x-2)^2 + 5 \geq 5 > 0$. **R1**

Therefore $f(x) > 0$ for all $x \in \mathbb{R}$. **AG**

Note: Award M1 for a clear attempt to complete the square with factor of 3 correctly handled. Award R1 for explicitly stating $(x-2)^2 \geq 0$ and concluding $f(x) \geq 5 > 0$. A purely discriminant-based argument ($\Delta = 144 - 204 < 0$ and leading coefficient > 0) earns M1 A1 R1 (3 marks only) unless accompanied by a conclusion earning A1.

Q19 — Four to the n Minus One Divisible by Three **MEDIUM**

(a) For $n = 3$: $4^3 - 1 = 64 - 1 = 63 = 3 \times 21$, which is divisible by 3. **A1**

(b) Write $4 = 3 + 1$, so $4^n = (3 + 1)^n$. **M1**

$$\text{Then } 4^n - 1 = (3 + 1)^n - 1.$$

$$\text{Alternatively, note that } 4^n - 1 = (4 - 1)(4^{n-1} + 4^{n-2} + \dots + 4 + 1) = 3(4^{n-1} + \dots + 1). \quad \textbf{M1}$$

Since $4^{n-1} + 4^{n-2} + \dots + 1$ is an integer, $4^n - 1 = 3 \times (\text{integer})$, so $4^n - 1$ is divisible by 3. **A1**
 Note: For part (b), award first M1 for writing $4^n - 1$ as $(4 - 1)(\dots)$ or recognising $4 \equiv 1 \pmod{3}$ leading to $4^n \equiv 1 \pmod{3}$. Award second M1 for completing the factorisation. Accept a modular arithmetic argument: $4 \equiv 1 \pmod{3} \Rightarrow 4^n \equiv 1^n = 1 \pmod{3} \Rightarrow 4^n - 1 \equiv 0 \pmod{3}$ for M1 M1 A1.

Q20 — Sum of Squares of Two Odd Integers Plus Two **MEDIUM**

Let $m = 2j + 1$ and $n = 2k + 1$ for integers j and k . **M1**
 Then $m^2 + n^2 + 2 = (2j + 1)^2 + (2k + 1)^2 + 2$
 $= (4j^2 + 4j + 1) + (4k^2 + 4k + 1) + 2$ **M1**
 $= 4j^2 + 4j + 4k^2 + 4k + 4$ **A1**
 $= 4(j^2 + j + k^2 + k + 1)$
 Since $j^2 + j + k^2 + k + 1$ is an integer, $m^2 + n^2 + 2$ is divisible by 4. **A1**
 Note: Award first M1 for defining odd integers algebraically as $2j + 1$ and $2k + 1$ (or using the same letter with a note). Award second M1 for expanding both squares. Final A1 requires the factored form $4(\dots)$ with an explicit divisibility conclusion.

Q21 — Divisibility of n^3 Minus n by Six **HARD**

METHOD 1
 Let n be a positive integer. Factor the expression: **M1**
 $n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1) = (n - 1)n(n + 1)$ **A1**
 This is the product of three consecutive integers. **R1**
 Among any three consecutive integers, at least one is divisible by 2 and exactly one is divisible by 3, so the product is divisible by $2 \times 3 = 6$. **M1**
 Therefore $(n - 1)n(n + 1)$ is divisible by 6 for all $n \in \mathbb{Z}^+$. **A1**
 Note: Award M1A1 for correct factorisation. Award R1 for explicitly identifying consecutive integers. Award M1 for the divisibility argument for both 2 and 3. Award final A1 for a complete conclusion. Parity argument alone (without factor of 3) earns at most 3 marks.

Q22 — Sum of Squares of Odd Integers **HARD**

Let $p = 2m + 1$ and $q = 2k + 1$ for some integers m, k . **M1**
 Then $p^2 + q^2 + 2 = (2m + 1)^2 + (2k + 1)^2 + 2$
 $= 4m^2 + 4m + 1 + 4k^2 + 4k + 1 + 2$ **A1**
 $= 4m^2 + 4m + 4k^2 + 4k + 4$ **A1**
 $= 4(m^2 + m + k^2 + k + 1)$ **M1**

Since $m, k \in \mathbb{Z}$, the expression $m^2 + m + k^2 + k + 1$ is an integer, so $p^2 + q^2 + 2 = 4(\dots)$ is divisible by 4. **A1**

Note: Award M1 for writing $p = 2m + 1$, $q = 2k + 1$ (same letter $m = k$ allowed). Award first A1 for correct expansion of both squares. Award second A1 for correct collection giving $+4$ as the constant. Award M1 for factoring out 4. Award final A1 for complete conclusion.

Q23 — Disproof by Counterexample for a Quadratic Prime Claim **HARD**

The statement is false. **A1**

A counterexample must be found. Testing $n = 41$: **M1**

$$f(41) = 41^2 - 41 + 41 = 41^2 = 1681 \quad \text{A1}$$

Since $1681 = 41^2$, it is divisible by 41 and greater than 41, so 1681 is not prime. **R1**

This is a valid counterexample, so the statement is false. **A1**

Note: Award A1 for stating the statement is false. Award M1 for a systematic attempt to find a counterexample. Award A1 for the correct value $f(41) = 1681$. Award R1 for explicitly stating why 1681 is not prime (it equals 41^2). Award final A1 for a clear conclusion linking the counterexample to the falsity. Other counterexamples (e.g. $n = 82$: $f(82) = 6683 = 41 \times 163$) are acceptable with correct computation.

Q24 — Positivity of a Quadratic Form **HARD**

METHOD 1 (Completing the square)

Attempt to complete the square in a : **M1**

$$a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 - \frac{b^2}{4} + b^2 \quad \text{A1}$$

$$= \left(a + \frac{b}{2}\right)^2 + \frac{3b^2}{4} \quad \text{A1}$$

Since $\left(a + \frac{b}{2}\right)^2 \geq 0$ and $\frac{3b^2}{4} \geq 0$, the sum is ≥ 0 . **M1**

Equality requires $a + \frac{b}{2} = 0$ and $b = 0$, giving $a = b = 0$. Since $a \neq b$ excludes $a = b = 0$, if $b = 0$ then

$$a \neq 0 \text{ so } \left(a + \frac{b}{2}\right)^2 = a^2 > 0. \text{ Therefore } a^2 + ab + b^2 > 0. \quad \text{A1}$$

Note: Award M1 for an attempt to complete the square. Award A1A1 for the correct completed square form. Award M1 for the non-negativity argument. Award final A1 for the strict inequality conclusion using $a \neq b$ (or a, b not both zero). Accept any equivalent valid argument for the strict inequality.

Q25 — An Inequality Claim by Aiko **HARD**

Aiko's claim is true. **A1**

Consider $(x + y)^2 - 4xy$: **M1**

$$(x + y)^2 - 4xy = x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2 = (x - y)^2 \quad \text{A1}$$

Since $(x - y)^2 \geq 0$ for all $x, y \in \mathbb{R}$, it follows that $(x + y)^2 \geq 4xy$. **M1**

Equality holds when $(x - y)^2 = 0$, i.e. when $x = y$. **A1**

Note: Award A1 for correctly identifying the claim as true. Award M1 for considering the difference $(x + y)^2 - 4xy$. Award A1 for correct simplification to $(x - y)^2$. Award M1 for the non-negativity of a square. Award final A1 for correctly stating the equality condition $x = y$.

Q26 — Parity of a Linear Expression **HARD**

The conjecture is true. **A1**

Let $n \in \mathbb{Z}^+$. Since $n \geq 1$, we have 2^n is even (as $2^n = 2 \cdot 2^{n-1}$ where $2^{n-1} \in \mathbb{Z}^+$). **M1**

Write $2^n = 2m$ for some integer m (specifically $m = 2^{n-1}$). **A1**

Then $2^n + 6 = 2m + 6 = 2(m + 3)$. **M1**

Since $m + 3 \in \mathbb{Z}$, the expression $2^n + 6$ is even for all $n \in \mathbb{Z}^+$. **A1**

Note: Award A1 for stating the conjecture is true. Award M1 for identifying 2^n as even with a reason. Award A1 for writing $2^n = 2m$ explicitly. Award M1 for factoring out 2. Award final A1 for a complete conclusion. A parity argument by cases (odd/even n) also earns full marks if complete.

Q27 — Telescoping Sum of Odd Integers **HARD**

$$(2n + 1)^2 - (2n - 1)^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) \quad \text{M1}$$

$$= 8n \quad \text{A1}$$

which is the shown result. **AG**

For the "hence" part, apply the same telescoping idea to consecutive integers $k - 1$ and k (rather than $2n - 1$ and $2n + 1$): for every integer $k \geq 1$, $k^2 - (k - 1)^2 = 2k - 1$. **M1**

Summing this identity from $k = 1$ to $k = N$, the left-hand side telescopes:

$$\sum_{k=1}^N [k^2 - (k - 1)^2] = N^2 - 0^2 = N^2,$$

and the right-hand side is $\sum_{k=1}^N (2k - 1)$, which is exactly $1 + 3 + 5 + \dots + (2N - 1)$, the sum of the first N positive odd integers. **A1**

Therefore the sum of the first N positive odd integers is N^2 . **R1**

Note: Award M1A1 AG for the algebraic expansion of the shown identity. The "hence" step requires the

telescoping identity $k^2 - (k - 1)^2 = 2k - 1$ (not the shown identity itself, which telescopes over odd-indexed terms two apart and does not directly give N^2); award M1 for stating this identity, A1 for the correct telescoping sum $\sum(k^2 - (k - 1)^2) = N^2$ matched to $\sum(2k - 1)$, and R1 for the concluding statement. Total: M1 A1 (AG, 0 marks) M1 A1 R1 = 5 marks.

Q28 — Inequality a over b Plus b over a with Equality **HARD**

Let $a, b \in \mathbb{R}^+$. Consider $\frac{a}{b} + \frac{b}{a} - 2$: **M1**

$$\frac{a}{b} + \frac{b}{a} - 2 = \frac{a^2 + b^2 - 2ab}{ab} = \frac{(a - b)^2}{ab}$$
A1

Since $a, b > 0$, we have $ab > 0$, and $(a - b)^2 \geq 0$. **M1**

Therefore $\frac{(a - b)^2}{ab} \geq 0$, which gives $\frac{a}{b} + \frac{b}{a} \geq 2$. **A1**

Equality holds when $(a - b)^2 = 0$, i.e. when $a = b$. **A1**

Note: Award M1 for considering the difference $\frac{a}{b} + \frac{b}{a} - 2$. Award A1 for correct algebraic simplification to $\frac{(a - b)^2}{ab}$. Award M1 for invoking positivity of denominator and non-negativity of numerator. Award A1 for concluding the inequality. Award final A1 for the equality condition $a = b$. A proof beginning from $(a - b)^2 \geq 0$ then dividing by $ab > 0$ is equally valid.

Q29 — Quadratic Positive Definite Condition **HARD**

For $f(x) = kx^2 - 4x + k > 0$ for all $x \in \mathbb{R}$, two conditions are required: **M1**

(i) $k > 0$ (parabola opens upward), and (ii) discriminant $\Delta < 0$. **A1**

Compute $\Delta = (-4)^2 - 4(k)(k) = 16 - 4k^2$. **A1**

Require $16 - 4k^2 < 0 \Rightarrow k^2 > 4 \Rightarrow k > 2$ or $k < -2$. **M1**

Combined with $k > 0$: the solution is $k > 2$. **A1**

Note: Award M1 for identifying both conditions needed (positive leading coefficient and negative discriminant). Award A1 for $k > 0$. Award A1 for correct discriminant $16 - 4k^2$. Award M1 for solving $\Delta < 0$ to obtain $k > 2$ or $k < -2$. Award final A1 for the correct intersection $k > 2$ only. Do not award final A1 if $k < -2$ is included or if the strict inequality is not satisfied. This is the root-rejection/validity trap: $k < -2$ satisfies $\Delta < 0$ but not $k > 0$.

Q30 — AM-GM Strict Inequality Proof **HARD**

Let $a, b \in \mathbb{R}^+$ with $a \neq b$. Consider

M1

$$\frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2}$$

A1

Since $a, b > 0$, both $\sqrt{a}$ and $\sqrt{b}$ are real and positive.

M1

Since $a \neq b$, we have $\sqrt{a} \neq \sqrt{b}$, so $(\sqrt{a}-\sqrt{b})^2 > 0$.

A1

Therefore $\frac{(\sqrt{a}-\sqrt{b})^2}{2} > 0$, which gives $\frac{a+b}{2} > \sqrt{ab}$.

A1

Note: Award M1 for considering the difference $\frac{a+b}{2} - \sqrt{ab}$. Award A1 for correctly writing the numerator as $(\sqrt{a}-\sqrt{b})^2$. Award M1 for invoking the positivity of a and b to ensure $\sqrt{a}, \sqrt{b}$ are real. Award A1 for using $a \neq b \Rightarrow \sqrt{a} \neq \sqrt{b}$ to obtain strict positivity. Award final A1 for the complete conclusion. The trap is failing to use $a \neq b$ to convert $\geq$ to $>$; this loses the final A1.