IB Math AA SL — Topic 1

Proof & Reasoning

Direct Algebraic Proof · Identities · Counterexamples

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Even integer:** $n = 2k, k \in \mathbb{Z}$; **Odd integer:** $n = 2k + 1, k \in \mathbb{Z}$
- **Consecutive integers:** $n,\ n+1,\ n+2,\ \ldots$, where $n \in \mathbb{Z}$
- **Difference of two squares:** $a^2 - b^2 = (a-b)(a+b)$
- **Completing the square:** $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$; expression is always positive iff $a > 0$ and discriminant $b^2 - 4ac < 0$
- **Divisibility form:** to show E is divisible by d, write $E = d \cdot$ (integer expression)
- **Disproof by counterexample:** find one specific n for which $P(n)$ is false and compute both sides explicitly to exhibit the contradiction
- **Sum/product parity rules:** odd $+$ odd $=$ even; odd $\times$ odd $=$ odd; even $\times$ anything $=$ even

GDC Tips

- **Integer table exploration:** use the Table function (TblStart $= 1$, ΔTbl $= 1$) to evaluate an expression for $n = 1, 2, \ldots, 10$ before conjecturing a divisibility pattern.
- **Counterexample search:** store a candidate expression as Y_1 and scroll the table to locate the first n for which the claim fails; record the numerical value as your counterexample.
- **Verify a specific case:** use the home-screen `STO→` key to assign $n = k$ and evaluate both sides of an identity to check a single substitution (verify, not prove).
- **Remainder check:** compute `remainder(expr, d)` or use `iPart` / `mod(expr, d)` to test divisibility for several values during exploration.
- **Expand & simplify on CAS:** use `expand()` and `factor()` on a CAS-enabled GDC to confirm algebraic manipulations; copy the exact form into written working.

Common Mistakes to Avoid

1. **Failing to define the integer variable.** Writing "let n be even" without stating $n = 2k$ for some $k \in \mathbb{Z}$ before any manipulation loses the opening mark; the proof has no declared object.
2. **Confusing verification with proof.** Checking that a statement holds for $n = 1, 2, 3$ is not a general proof; a "show that" or "prove" command requires an algebraic argument valid for all integers, not a numerical check.
3. **Assuming the target in a "show that" question.** Beginning with the printed result and working backwards (or writing both sides and equating them) is circular reasoning; always start from one side and reach the other by a forward chain.
4. **Incomplete counterexample.** Stating only the value of n without computing both sides of the claimed inequality or identity and explicitly identifying the contradiction does not earn the mark; the contradiction must be stated in words.
5. **Omitting the conclusion line.** Every proof must end with a sentence restating the claim (e.g. "therefore $n^2 - n$ is always even"); stopping at the factored expression without the conclusion loses the final reasoning mark R1.
6. **Wrong completing-the-square sign decision.** When showing an expression is always positive, students write $a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$ but then claim the constant term is positive without checking $b^2 < 4ac$; the sign of the discriminant must be verified explicitly to justify the conclusion.

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Section A — Easy

EASY

1. Let n be an integer. Show that the sum of n and $n+2$ is always even. *Calculator not allowed* **[3 marks]**

2. Let n be an integer. Show that the product of two consecutive integers, n and $n+1$, is always even. *Calculator not allowed* **[3 marks]**

3. Consider the statement: "For all integers n, the expression n^2+n+1 is prime." Disprove this statement by finding a counterexample, where $n \in \mathbb{Z}^+$. *Calculator allowed* **[3 marks]**

4. Let n be an integer. Show that n^2-n is always even. *Calculator not allowed* **[3 marks]**

5. Determine whether the following statement is true or false: "The sum of any three consecutive integers is divisible by 3." Justify your answer. *Calculator allowed* **[3 marks]**

6. The following table shows a numerical exploration of the expression $n^2 - 1$ for positive integer values of n, where $n \geq 2$.

n	2	3	4	5
$n^2 - 1$	3	8	15	24
$(n-1)(n+1)$	3	8	15	24

Using the table as inspiration, show that $n^2 - 1 = (n-1)(n+1)$ for all integers n. *Calculator allowed* **[3 marks]**

7. Consider the statement: "For all real numbers x, $x^2 \geq x$." Disprove this statement by finding a counterexample, where $x \in \mathbb{R}$. *Calculator allowed* **[3 marks]**

8. Let n be an integer. Show that the product of two odd integers, $2n+1$ and $2n+3$, is always odd. *Calculator not allowed* **[3 marks]**

9. Consider the expression $x^2 + 6x + 13$, where $x \in \mathbb{R}$. By completing the square, show that $x^2 + 6x + 13 > 0$ for all real values of x. *Calculator allowed* **[3 marks]**

10. Verify that $n = 4$ satisfies the equation $n^2 - 5n + 4 = 0$. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. Let n be an integer. Show that the product of any two consecutive even integers is divisible by 8. *Calculator not allowed* **[4 marks]**

12. Let n be an integer. Prove that $n^2 + n$ is always even. *Calculator allowed* **[4 marks]**

13. Consider the statement: "For all integers n, if n^2 is divisible by 4, then n is divisible by 4."
Disprove this statement by finding a counterexample. Justify your answer. *Calculator allowed* **[4 marks]**

14. Let n be an integer. Prove that the difference of the squares of any two consecutive odd integers is divisible by 8. *Calculator not allowed* **[4 marks]**

15. The table below shows the values of n and $f(n) = n^2 - n + 11$ for $n \in \{1, 2, 3, 4, 5\}$.

n	1	2	3	4	5
$f(n)$	11	13	17	23	31

A student conjectures that $f(n) = n^2 - n + 11$ is prime for all positive integers n.

(a) Verify the conjecture for $n = 5$. **[1 marks]**

(b) Disprove the conjecture by finding a value of n for which $f(n)$ is not prime. **[3 marks]**

16. Let a and b be real numbers where $a > 0$ and $b > 0$. Show that $\frac{a}{b} + \frac{b}{a} \geq 2$. *Calculator not allowed* **[4 marks]**

17. Let n be an integer. Prove that $(n+3)^2 - (n-1)^2$ is always divisible by 8. *Calculator allowed* **[4 marks]**

18. Consider the expression $f(x) = 3x^2 - 12x + 17$, where $x \in \mathbb{R}$.
Show that $f(x) > 0$ for all real values of x. *Calculator not allowed* **[4 marks]**

19. Let n be a positive integer. Consider the statement: "$4^n - 1$ is divisible by 3 for all positive integers n."

(a) Verify the statement for $n = 3$. **[1 marks]**

(b) Prove the statement by writing $4^n - 1$ in a suitable factored or expanded form. **[3 marks]**

Calculator allowed **[4 marks]**

20. Let m and n be integers. Prove that if m is odd and n is odd, then $m^2 + n^2 + 2$ is divisible by 4. *Calculator allowed*
[4 marks]

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Section C — Hard

HARD

21. Let $n \in \mathbb{Z}^+$. Show that $n^3 - n$ is divisible by 6 for all positive integers n. *Calculator not allowed* **[5 marks]**

22. Let p and q be odd integers. Prove that $p^2 + q^2 + 2$ is divisible by 4. *Calculator not allowed* **[5 marks]**

23. Consider the statement: "For all integers n where $n \geq 2$, the expression $n^2 - n + 41$ is prime." Determine, with full justification, whether this statement is true or false. *Calculator allowed* **[5 marks]**

24. Let $a, b \in \mathbb{R}$ with $a \neq b$. Show that $a^2 + ab + b^2 > 0$. *Calculator not allowed* **[5 marks]**

25. Aiko claims that for all real numbers x and y, $(x+y)^2 \geq 4xy$. Determine whether Aiko's claim is true or false. If true, prove the result. If false, provide a counterexample and state the correct condition on x and y under which equality holds. *Calculator allowed* **[5 marks]**

26. The following table shows values of $f(n) = 2^n + 6$ for selected positive integers n.

n	1	2	3	4
$f(n)$	8	10	14	22

A student conjectures that $f(n) = 2^n + 6$ is always even for $n \in \mathbb{Z}^+$. Determine whether this conjecture is true, and prove your conclusion. *Calculator allowed* **[5 marks]**

27. Let n be an integer with $n \geq 1$. Show that $(2n+1)^2 - (2n-1)^2 = 8n$, and hence prove that the sum of the first N positive odd integers is N^2, where $N \in \mathbb{Z}^+$. *Calculator not allowed* **[5 marks]**

28. Let a and b be positive real numbers. Prove that $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$, stating clearly when equality holds. *Calculator not allowed* **[5 marks]**

29. Consider the quadratic expression $f(x) = kx^2 - 4x + k$ where $k \in \mathbb{R}$, $k \neq 0$. Find the values of k for which $f(x) > 0$ for all $x \in \mathbb{R}$, justifying your answer fully. *Calculator allowed* **[5 marks]**

30. Prove that for any two distinct real numbers a and b, the arithmetic mean of a and b is strictly greater than their geometric mean when both a and b are positive and $a \neq b$. That is, prove that $\dfrac{a+b}{2} > \sqrt{ab}$ for all $a, b \in \mathbb{R}^+$ with $a \neq b$. *Calculator not allowed* **[5 marks]**

Worked Solutions

Q1 — Sum of two integers two apart is even [EASY]

Let n be an integer, so the two integers are n and $n+2$. **M1**

Their sum is $n+(n+2) = 2n+2 = 2(n+1)$. **A1**

Since $2(n+1)$ is a multiple of 2, the sum is always even. **R1**

Note: The conclusion must be stated in words or by reference to the factor of 2; accept equivalent phrasing.

Q2 — Product of consecutive integers is even [EASY]

Let n be an integer, so the consecutive integers are n and $n+1$. **M1**

Either n or $n+1$ must be even (consecutive integers have opposite parity), so their product $n(n+1)$ contains a factor of 2. **A1**

Hence $n(n+1)$ is always even. **R1**

METHOD 2

$n(n+1) = n^2+n$. **M1**

Consider cases: if $n = 2k$ then $n(n+1) = 2k(2k+1) = 2(k(2k+1))$, even; if $n = 2k+1$ then $n(n+1) = (2k+1)(2k+2) = 2(2k+1)(k+1)$, even. **A1**

Hence $n(n+1)$ is always even. **R1**

Q3 — Disproof by counterexample for prime claim [EASY]

Test $n = 3$: **M1**

$n^2+n+1 = 9+3+1 = 13$, which is prime.

Test $n = 4$: $16+4+1 = 21 = 3\times 7$, which is not prime. **A1**

Since $n = 4$ gives a non-prime value, the statement is false. **R1**

Note: Any valid counterexample earns full marks; accept $n = 7$ ($57 = 3\times 19$), $n = 10$ ($111 = 3\times 37$), etc. The contradiction must be stated.

Q4 — $n^2 - n$ is always even [EASY]

Let n be an integer. Write $n^2 - n = n(n-1)$. **M1**

The integers $n-1$ and n are consecutive, so one of them is even, giving $n(n-1)$ a factor of 2. **A1**

Hence $n^2 - n$ is always even. **R1**

Q5 — Sum of three consecutive integers divisible by 3 [EASY]

Let the three consecutive integers be n, $n+1$, and $n+2$, where n is an integer. **M1**

Their sum is $n+(n+1)+(n+2)=3n+3=3(n+1)$. **A1**

Since $3(n+1)$ is a multiple of 3, the statement is **true** and the sum is always divisible by 3. **R1**

Note: The word "true" or an equivalent verdict must appear; award R1 only if a correct general algebraic argument is present.

Q6 — Algebraic identity $n^2-1=(n-1)(n+1)$ [EASY]

Expand the right-hand side: **M1**

$(n-1)(n+1)=n^2+n-n-1$ **A1**

$=n^2-1$ **AG**

which is the left-hand side, hence the identity holds for all integers n. **R1**

Note: Final line is AG and carries no marks; R1 is awarded for a statement that the identity has been established in general (not just for the table values).

Q7 — Disproof of $x^2 \geq x$ for all reals [EASY]

Test $x=0.5$: **M1**

$x^2=0.25$ and $x=0.5$, so $x^2=0.25<0.5=x$. **A1**

Since $x^2<x$ when $x=0.5$, the statement is false. **R1**

Note: Any $x\in(0,1)$ is a valid counterexample; the computed values must be shown and the contradiction stated explicitly.

Q8 — Product of two odd integers is odd [EASY]

Let $2n+1$ and $2n+3$ be two odd integers, where n is an integer. **M1**

$(2n+1)(2n+3)=4n^2+6n+2n+3=4n^2+8n+3=2(2n^2+4n+1)+1$ **A1**

Since this is of the form $2k+1$ where $k=2n^2+4n+1$ is an integer, the product is always odd. **R1**

Note: Accept any correct factored form that clearly identifies an odd structure; the conclusion must be stated.

Q9 — Completing the square to show expression positive [EASY]

Attempt to complete the square: **M1**

$x^2+6x+13=(x+3)^2-9+13=(x+3)^2+4$ **A1**

Since $(x+3)^2\geq 0$ for all $x\in\mathbb{R}$, we have $(x+3)^2+4\geq 4>0$, so $x^2+6x+13>0$ for all real x. **R1**

Note: The inequality chain and conclusion must appear; do not award R1 for merely stating the completed-square form without justification.

Q10 — Verify $n = 4$ satisfies quadratic [EASY]

Substitute $n = 4$ into $n^2 - 5n + 4$: **M1**

$(4)^2 - 5(4) + 4 = 16 - 20 + 4$ **A1**

$= 0$ **A1**

Since the result equals 0, $n = 4$ satisfies the equation.

Note: All three arithmetic steps must be shown; the conclusion that the result equals 0 must be stated.

Q11 — Product of consecutive even integers divisible by 8 [MEDIUM]

Let n be an integer. Then two consecutive even integers can be written as $2n$ and $2n + 2$. **M1**

Their product is $2n(2n + 2) = 4n(n + 1)$. **A1**

Since n and $n + 1$ are consecutive integers, one of them must be even, so $n(n + 1)$ is divisible by 2. **M1**

Therefore $4n(n + 1) = 4 \times 2k = 8k$ for some integer k, which is divisible by 8. **A1**

Note: Award M1 for defining consecutive even integers algebraically as $2n$ and $2n + 2$ (or $2k$ and $2k + 2$). Accept equivalent notation. The conclusion must state divisibility by 8 explicitly.

Q12 — $n^2 + n$ is always even [MEDIUM]

Let n be an integer. Then $n^2 + n = n(n + 1)$. **M1**

Since n and $n + 1$ are consecutive integers, exactly one of them is even. **R1**

Therefore their product $n(n + 1)$ is even, i.e. $n^2 + n$ is always even. **A1**

METHOD 2

Case 1: Let $n = 2k$. Then $n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$, which is even.
Case 2: Let $n = 2k + 1$. Then $n^2 + n = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$, which is even. **M1 A1**

In both cases $n^2 + n$ is even. **R1**

Note: Both methods earn 3 marks total (M1, R1, A1). Award M1 for a valid attempt to factor or use parity cases. Final conclusion required for A1.

Q13 — Disproof by counterexample: n^2 divisible by 4 [MEDIUM]

Consider $n = 6$. **M1**

Then $n^2 = 36 = 4 \times 9$, so n^2 is divisible by 4. **A1**

However, $6 \div 4 = 1.5$, so 6 is not divisible by 4. **A1**

This is a counterexample, so the statement is false. **R1**

Note: Accept any even integer n not divisible by 4, e.g. $n = 2$: $n^2 = 4$ is divisible by 4 but 2 is not divisible by 4. Award R1 only if contradiction is explicitly stated. Do not award R1 for a statement without computation.

Q14 — Difference of squares of consecutive odd integers divisible by 8 [MEDIUM]

Let n be an integer. Two consecutive odd integers can be written as $2n + 1$ and $2n + 3$. **M1**

Their difference of squares is $(2n + 3)^2 - (2n + 1)^2$.

$= (4n^2 + 12n + 9) - (4n^2 + 4n + 1)$ **M1**

$= 8n + 8 = 8(n + 1)$ **A1**

Since $n + 1$ is an integer, $8(n + 1)$ is divisible by 8. **A1**

Note: Award first M1 for defining consecutive odd integers algebraically. Award second M1 for expanding both squares. Final A1 requires the factored form $8(\dots)$ and a conclusion.

Q15 — Conjecture about $n^2 - n + 11$ being prime [MEDIUM]

(a) $f(5) = 25 - 5 + 11 = 31$, which is prime. **A1**

(b) Consider $n = 11$: **M1**

$f(11) = 121 - 11 + 11 = 121 = 11^2$ **A1**

Since $121 = 11 \times 11$ is not prime, the conjecture is disproved. **A1**

Note: For part (b), award M1 for a systematic attempt to find a counterexample (e.g. testing multiples of 11 or values where $n^2 - n + 11$ shares a factor). Accept any valid counterexample with correct computation and explicit statement that the result is not prime. $n = 11$ is the smallest counterexample.

Q16 — Show $\frac{a}{b} + \frac{b}{a} \geq 2$ for $a, b > 0$ [MEDIUM]

Consider $\dfrac{a}{b} + \dfrac{b}{a} - 2 = \dfrac{a^2 + b^2 - 2ab}{ab}$. **M1**

$= \dfrac{(a - b)^2}{ab}$ **A1**

Since $a > 0$ and $b > 0$, we have $ab > 0$, and $(a - b)^2 \geq 0$ for all real a, b. **R1**

Therefore $\dfrac{(a-b)^2}{ab} \geq 0$, which gives $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$. **A1**

Note: Award M1 for combining the fractions over a common denominator ab. Award R1 for explicitly stating both $(a-b)^2 \geq 0$ and $ab > 0$. Final A1 requires a conclusion restating the inequality. Do not award A1 if the direction of argument is reversed (assumes the result).

Q17 — $(n+3)^2 - (n-1)^2$ divisible by 8 [MEDIUM]

$(n+3)^2 - (n-1)^2 = (n^2 + 6n + 9) - (n^2 - 2n + 1)$ **M1**

$= 8n + 8$ **A1**

$= 8(n+1)$ **A1**

Since $n+1$ is an integer, $8(n+1)$ is divisible by 8. **R1**

METHOD 2

Using difference of two squares: $(n+3)^2 - (n-1)^2 = [(n+3) + (n-1)][(n+3) - (n-1)]$ **M1**

$= (2n+2)(4) = 8(n+1)$ **A1 A1**

Since $n+1$ is an integer, this is divisible by 8. **R1**

Note: Award M1 for expanding or factoring correctly. Award R1 only if a conclusion about divisibility by 8 is explicitly stated. Both methods earn 4 marks.

Q18 — Show $f(x) = 3x^2 - 12x + 17 > 0$ for all $x \in \mathbb{R}$ [MEDIUM]

Attempt to complete the square: $3x^2 - 12x + 17 = 3(x^2 - 4x) + 17$ **M1**

$= 3(x-2)^2 - 12 + 17$ **A1**

$= 3(x-2)^2 + 5$ **A1**

Since $(x-2)^2 \geq 0$ for all $x \in \mathbb{R}$, we have $3(x-2)^2 + 5 \geq 5 > 0$. **R1**

Therefore $f(x) > 0$ for all $x \in \mathbb{R}$. **AG**

Note: Award M1 for a clear attempt to complete the square with factor of 3 correctly handled. Award R1 for explicitly stating $(x-2)^2 \geq 0$ and concluding $f(x) \geq 5 > 0$. A purely discriminant-based argument ($\Delta = 144 - 204 < 0$ and leading coefficient > 0) earns M1 A1 R1 (3 marks only) unless accompanied by a conclusion earning A1.

Q19 — $4^n - 1$ divisible by 3 [MEDIUM]

(a) For $n = 3$: $4^3 - 1 = 64 - 1 = 63 = 3 \times 21$, which is divisible by 3. **A1**

(b) Write $4 = 3 + 1$, so $4^n = (3+1)^n$. **M1**

Then $4^n - 1 = (3+1)^n - 1$.

Alternatively, note that $4^n - 1 = (4-1)(4^{n-1} + 4^{n-2} + \cdots + 4 + 1) = 3(4^{n-1} + \cdots + 1)$. **M1**

Since $4^{n-1} + 4^{n-2} + \cdots + 1$ is an integer, $4^n - 1 = 3 \times (\text{integer})$, so $4^n - 1$ is divisible by 3. **A1**

Note: For part (b), award first M1 for writing $4^n - 1$ as $(4-1)(\dots)$ or recognising $4 \equiv 1 \pmod 3$ leading to $4^n \equiv 1 \pmod 3$. Award second M1 for completing the factorisation. Accept a modular arithmetic argument: $4 \equiv 1 \pmod 3 \Rightarrow 4^n \equiv 1^n = 1 \pmod 3 \Rightarrow 4^n - 1 \equiv 0 \pmod 3$ for M1 M1 A1.

Q20 — $m^2 + n^2 + 2$ divisible by 4 for odd m, n [MEDIUM]

Let $m = 2j + 1$ and $n = 2k + 1$ for integers j and k. **M1**

Then $m^2 + n^2 + 2 = (2j+1)^2 + (2k+1)^2 + 2$

$= (4j^2 + 4j + 1) + (4k^2 + 4k + 1) + 2$ **M1**

$= 4j^2 + 4j + 4k^2 + 4k + 4$ **A1**

$= 4(j^2 + j + k^2 + k + 1)$

Since $j^2 + j + k^2 + k + 1$ is an integer, $m^2 + n^2 + 2$ is divisible by 4. **A1**

Note: Award first M1 for defining odd integers algebraically as $2j+1$ and $2k+1$ (or using the same letter with a note). Award second M1 for expanding both squares. Final A1 requires the factored form $4(\dots)$ with an explicit divisibility conclusion.

Q21 — Divisibility of $n^3 - n$ by 6 [HARD]

METHOD 1

Let n be a positive integer. Factor the expression: **M1**

$n^3 - n = n(n^2 - 1) = n(n-1)(n+1) = (n-1)n(n+1)$ **A1**

This is the product of three consecutive integers. **R1**

Among any three consecutive integers, at least one is divisible by 2 and exactly one is divisible by 3, so the product is divisible by $2 \times 3 = 6$. **M1**

Therefore $(n-1)n(n+1)$ is divisible by 6 for all $n \in \mathbb{Z}^+$. **A1**

Note: Award M1A1 for correct factorisation. Award R1 for explicitly identifying consecutive integers. Award M1 for the divisibility argument for both 2 and 3. Award final A1 for a complete conclusion. Parity argument alone (without factor of 3) earns at most 3 marks.

Q22 — Sum of squares of odd integers [HARD]

Let $p = 2m + 1$ and $q = 2k + 1$ for some integers m, k. **M1**

Then $p^2 + q^2 + 2 = (2m+1)^2 + (2k+1)^2 + 2$

$= 4m^2 + 4m + 1 + 4k^2 + 4k + 1 + 2$ **A1**

$= 4m^2 + 4m + 4k^2 + 4k + 4$ **A1**

$= 4(m^2 + m + k^2 + k + 1)$ **M1**

Since $m, k \in \mathbb{Z}$, the expression $m^2 + m + k^2 + k + 1$ is an integer, so $p^2 + q^2 + 2 = 4(\ldots)$ is divisible by 4. **A1**

Note: Award M1 for writing $p = 2m + 1$, $q = 2k + 1$ (same letter $m = k$ allowed). Award first A1 for correct expansion of both squares. Award second A1 for correct collection giving $+4$ as the constant. Award M1 for factoring out 4. Award final A1 for complete conclusion.

Q23 — Disproof by counterexample [HARD]

The statement is **false**. **A1**

A counterexample must be found. Testing $n = 41$: **M1**

$f(41) = 41^2 - 41 + 41 = 41^2 = 1681$ **A1**

Since $1681 = 41^2$, it is divisible by 41 and greater than 41, so 1681 is **not prime**. **R1**

This is a valid counterexample, so the statement is false. **A1**

Note: Award A1 for stating the statement is false. Award M1 for a systematic attempt to find a counterexample. Award A1 for the correct value $f(41) = 1681$. Award R1 for explicitly stating why 1681 is not prime (it equals 41^2). Award final A1 for a clear conclusion linking the counterexample to the falsity. Other counterexamples (e.g. $n = 82$: $f(82) = 6683 = 41 \times 163$) are acceptable with correct computation.

Q24 — Positivity of $a^2 + ab + b^2$ [HARD]

METHOD 1 (Completing the square)

Attempt to complete the square in a: **M1**

$a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^2 - \frac{b^2}{4} + b^2$ **A1**

$= \left(a + \frac{b}{2}\right)^2 + \frac{3b^2}{4}$ **A1**

Since $\left(a + \frac{b}{2}\right)^2 \geq 0$ and $\frac{3b^2}{4} \geq 0$, the sum is ≥ 0. **M1**

Equality requires $a + \frac{b}{2} = 0$ and $b = 0$, giving $a = b = 0$. Since $a \neq b$ is not sufficient to exclude both zero simultaneously when they are equal, but since we also require $a \neq b$, if $b = 0$ then $a \neq 0$ so $\left(a + \frac{b}{2}\right)^2 = a^2 > 0$. Therefore $a^2 + ab + b^2 > 0$. **A1**

Note: Award M1 for an attempt to complete the square. Award A1A1 for the correct completed square form. Award M1 for the non-negativity argument. Award final A1 for the strict inequality conclusion using $a \neq b$ (or a, b not both zero). Accept any equivalent valid argument for the strict inequality.

Q25 — AM-GM inequality $(x+y)^2 \geq 4xy$ [HARD]

Aiko's claim is **true**. **A1**

Consider $(x+y)^2 - 4xy$: **M1**

$(x+y)^2 - 4xy = x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2 = (x-y)^2$ **A1**

Since $(x-y)^2 \geq 0$ for all $x, y \in \mathbb{R}$, it follows that $(x+y)^2 \geq 4xy$. **M1**

Equality holds when $(x-y)^2 = 0$, i.e. when $x = y$. **A1**

Note: Award A1 for correctly identifying the claim as true. Award M1 for considering the difference $(x+y)^2 - 4xy$. Award A1 for correct simplification to $(x-y)^2$. Award M1 for the non-negativity of a square. Award final A1 for correctly stating the equality condition $x = y$.

Q26 — Parity of $2^n + 6$ [HARD]

The conjecture is **true**. **A1**

Let $n \in \mathbb{Z}^+$. Since $n \geq 1$, we have 2^n is even (as $2^n = 2 \cdot 2^{n-1}$ where $2^{n-1} \in \mathbb{Z}^+$). **M1**

Write $2^n = 2m$ for some integer m (specifically $m = 2^{n-1}$). **A1**

Then $2^n + 6 = 2m + 6 = 2(m+3)$. **M1**

Since $m + 3 \in \mathbb{Z}$, the expression $2^n + 6$ is even for all $n \in \mathbb{Z}^+$. **A1**

Note: Award A1 for stating the conjecture is true. Award M1 for identifying 2^n as even with a reason. Award A1 for writing $2^n = 2m$ explicitly. Award M1 for factoring out 2. Award final A1 for a complete conclusion. A parity argument by cases (odd/even n) also earns full marks if complete.

Q27 — Telescoping sum of odd integers [HARD]

$(2n+1)^2 - (2n-1)^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1)$ **M1**

$= 8n$ **A1**

AG

The n-th positive odd integer is $2n - 1$. Sum the telescoping identity from $n = 1$ to $n = N$: **M1**

$$\sum_{n=1}^{N} [(2n+1)^2 - (2n-1)^2] = \sum_{n=1}^{N} 8n$$

The left side telescopes: $(2N+1)^2 - 1^2 = 8\sum_{n=1}^{N} n$

Alternatively, summing $8n$ the identity gives $\sum_{n=1}^{N}(2n-1) = N^2$ after rearrangement. **A1**

More directly: let $S = \sum_{n=1}^{N}(2n-1)$. From the shown result, $(2n+1)^2 = (2n-1)^2 + 8n$, so writing the odd integers as $1, 3, 5, \ldots, 2N-1$: their sum telescopes to N^2. **A1**

Note: Award M1A1 AG for the algebraic expansion. Award M1 for a correct telescoping or inductive strat-

egy using the result. Award A1A1 for a complete valid proof of the sum formula. Accept a full induction proof for the sum result after the AG for M1A1A1.

Q28 — $a/b + b/a \geq 2$ with equality [HARD]

Let $a, b \in \mathbb{R}^+$. Consider $\dfrac{a}{b} + \dfrac{b}{a} - 2$: **M1**

$\dfrac{a}{b} + \dfrac{b}{a} - 2 = \dfrac{a^2 + b^2 - 2ab}{ab} = \dfrac{(a-b)^2}{ab}$ **A1**

Since $a, b > 0$, we have $ab > 0$, and $(a-b)^2 \geq 0$. **M1**

Therefore $\dfrac{(a-b)^2}{ab} \geq 0$, which gives $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$. **A1**

Equality holds when $(a-b)^2 = 0$, i.e. when $a = b$. **A1**

Note: Award M1 for considering the difference $\frac{a}{b} + \frac{b}{a} - 2$. Award A1 for correct algebraic simplification to $\frac{(a-b)^2}{ab}$. Award M1 for invoking positivity of denominator and non-negativity of numerator. Award A1 for concluding the inequality. Award final A1 for the equality condition $a = b$. A proof beginning from $(a-b)^2 \geq 0$ then dividing by $ab > 0$ is equally valid.

Q29 — Quadratic positive definite condition [HARD]

For $f(x) = kx^2 - 4x + k > 0$ for all $x \in \mathbb{R}$, two conditions are required: **M1**

(i) $k > 0$ (parabola opens upward), and (ii) discriminant $\Delta < 0$. **A1**

Compute $\Delta = (-4)^2 - 4(k)(k) = 16 - 4k^2$. **A1**

Require $16 - 4k^2 < 0 \Rightarrow k^2 > 4 \Rightarrow k > 2$ or $k < -2$. **M1**

Combined with $k > 0$: the solution is $k > 2$. **A1**

Note: Award M1 for identifying both conditions needed (positive leading coefficient and negative discriminant). Award A1 for $k > 0$. Award A1 for correct discriminant $16 - 4k^2$. Award M1 for solving $\Delta < 0$ to obtain $k > 2$ or $k < -2$. Award final A1 for the correct intersection $k > 2$ only. Do not award final A1 if $k < -2$ is included or if the strict inequality is not satisfied. This is the root-rejection/validity trap: $k < -2$ satisfies $\Delta < 0$ but not $k > 0$.

Q30 — AM-GM strict inequality proof [HARD]

Let $a, b \in \mathbb{R}^+$ with $a \neq b$. Consider **M1**

$\dfrac{a+b}{2} - \sqrt{ab} = \dfrac{a + b - 2\sqrt{ab}}{2} = \dfrac{(\sqrt{a} - \sqrt{b})^2}{2}$ **A1**

Since $a, b > 0$, both $\sqrt{a}$ and $\sqrt{b}$ are real and positive. **M1**

Since $a \neq b$, we have $\sqrt{a} \neq \sqrt{b}$, so $(\sqrt{a} - \sqrt{b})^2 > 0$. **A1**

Therefore $\dfrac{(\sqrt{a}-\sqrt{b})^2}{2} > 0$, which gives $\dfrac{a+b}{2} > \sqrt{ab}$. **A1**

Note: Award M1 for considering the difference $\frac{a+b}{2} - \sqrt{ab}$. Award A1 for correctly writing the numerator as $(\sqrt{a}-\sqrt{b})^2$. Award M1 for invoking the positivity of a and b to ensure $\sqrt{a}, \sqrt{b}$ are real. Award A1 for using $a \neq b \Rightarrow \sqrt{a} \neq \sqrt{b}$ to obtain strict positivity. Award final A1 for the complete conclusion. The trap is failing to use $a \neq b$ to convert $\geq$ to $>$; this loses the final A1.