IB Math AA SL — Topic 2

Functions Toolkit

Function Language · Composite & Inverse Functions · Key Features & Intersections

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:____________________ Date:____________

Score:________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Composite function:** $(g \circ f)(x) = g(f(x))$; evaluate the *inner* function first.
- **Inverse function condition:** $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ for all x in the appropriate domain.
- **Domain of inverse:** domain of f^{-1} = range of f; range of f^{-1} = domain of f.
- **Horizontal-line test:** f has an inverse function if and only if every horizontal line meets the graph at most once (one-to-one).
- **Graph of inverse:** $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.
- **Self-inverse check:** f is self-inverse if $f(f(x)) = x$, equivalently $f^{-1}(x) = f(x)$.
- **Intersection condition:** $f(x) = g(x)$ gives the x-coordinates of intersection points; substitute back to find y-coordinates.

GDC Tips

- **Graphing and tracing:** Enter $f(x)$ in the `Y=` editor; use `TRACE` or `TABLE` to read off function values, intercepts, and approximate domain/range.
- **Finding intersections:** Graph both $y = f(x)$ and $y = g(x)$, then use `2nd CALC` $\to$ `5:intersect`; move the cursor near each crossing point and confirm with `ENTER` three times. Record *all* intersection points.
- **Locating maxima, minima, and zeros:** Use `2nd CALC` $\to$ `2:zero`, `3:minimum`, or `4:maximum`; set left/right bounds bracketing the feature before pressing `ENTER`.
- **Evaluating a composite numerically:** Store the inner value with `STO`$\to$, then evaluate the outer function, *or* nest expressions directly: `g(f(2))` typed as a single expression on the home screen.
- **Checking an inverse:** Graph $f(x)$, $f^{-1}(x)$ (enter its algebraic form), and $y = x$ together; the two curves should appear as reflections across $y = x$. Use `ZSquare` (`ZOOM 5`) so the reflection looks correct.
- **Solving $f(x) = k$ numerically:** Enter $y_1 = f(x)$ and $y_2 = k$ (a horizontal line), then use `intersect` to find *every* solution on the stated domain — scroll the window to check for additional crossings.

Common Mistakes to Avoid

1. **Reversing composite order.** Writing $(g \circ f)(x) = f(g(x))$ instead of $g(f(x))$. The notation $g \circ f$ means *apply f first, then g*. Always identify the inner and outer functions before substituting.
2. **Forgetting to state the domain of an inverse.** Finding $f^{-1}(x)$ algebraically but omitting its domain. Since the domain of f^{-1} equals the range of f, you must determine and state it explicitly; an unrestricted answer loses the final mark.
3. **Keeping both branches when a condition eliminates one.** When solving for a parameter in a composite and obtaining $\pm$ answers, failing to reject the branch excluded by a given constraint (e.g. $x > 0$, $k \in \mathbb{Z}^+$, or $|r| < 1$). Always check every solution against the stated domain or parameter set.
4. **Finding only one intersection point.** Using `intersect` on the GDC and stopping after the first crossing. Many pairs of functions meet at two or more points; scroll the viewing window and repeat `intersect` for every crossing on the stated domain.
5. **Misreading graph-reading questions.** Confusing $f^{-1}(c)$, $(f \circ f)(c)$, and $f(c)$ when reading from a drawn graph. For $f^{-1}(c)$: find c on the y-axis and read across to the curve, then down to the x-axis. For $(f \circ f)(c)$: evaluate $f(c)$ first, use that output as the new input. Only use labelled features; do not interpolate unlabelled points.
6. **Treating a non-one-to-one function as invertible over its full domain.** Applying the inverse-finding procedure to a function such as $f(x) = x^2$ over $\mathbb{R}$ without first restricting the domain to make it one-to-one. The inverse exists only on the restricted domain; the answer must reflect this restriction explicitly.

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Section A — Easy

EASY

1. Let $f(x) = 3x - 5$, for $x \in \mathbb{R}$.

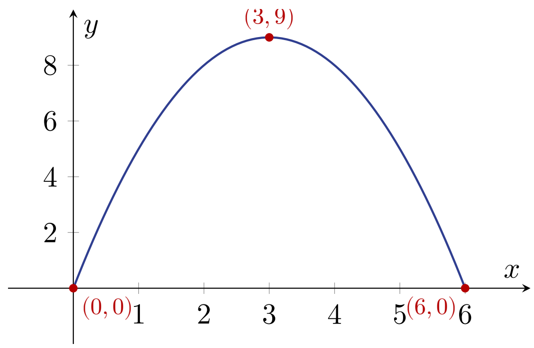
(a) Write down $f(0)$. **[1 marks]**

(b) Find $f^{-1}(x)$. **[2 marks]**

Calculator not allowed **[3 marks]**

2. Let $f(x) = x^2 + 1$ and $g(x) = 2x - 3$, for $x \in \mathbb{R}$.
Find $(g \circ f)(x)$. *Calculator not allowed* **[3 marks]**

3. The graph of $y = f(x)$ is shown below.



(a) Write down $f(3)$. **[1 marks]**

(b) Write down the range of f for $0 \leq x \leq 6$. **[2 marks]**

Calculator allowed **[3 marks]**

4. Let $f(x) = \dfrac{x+4}{2}$, for $x \in \mathbb{R}$.

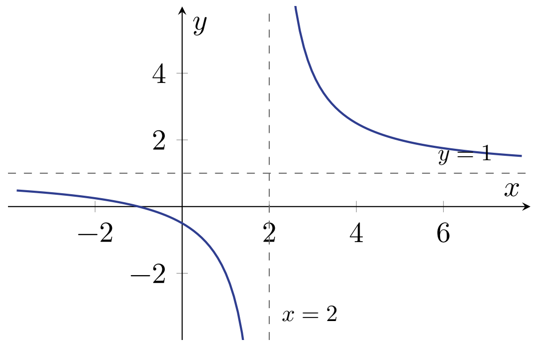
(a) Find $f^{-1}(x)$. **[2 marks]**

(b) Write down $f^{-1}(3)$. **[1 marks]**

Calculator not allowed **[3 marks]**

5. Let $f(x) = 2x + 1$ and $g(x) = x^2$, for $x \in \mathbb{R}$.
Find $(f \circ g)(3)$. *Calculator allowed* **[3 marks]**

6. Let $f(x) = \dfrac{3}{x-2} + 1$, for $x \in \mathbb{R}, x \neq 2$.
The graph of f is shown below.



(a) Write down the equation of the vertical asymptote of f. **[1 marks]**

(b) Write down the equation of the horizontal asymptote of f. **[1 marks]**

(c) Find $f(5)$. **[1 marks]**

Calculator allowed **[3 marks]**

7. Let $f(x) = 5x - 2$ and $g(x) = 3x + k$, for $x \in \mathbb{R}$, where $k \in \mathbb{R}$.
Given that $(f \circ g)(1) = 18$, find the value of k. *Calculator allowed* **[3 marks]**

8. Let $f(x) = x^2 - 4$, for $x \in \mathbb{R}$.

(a) Write down the y-intercept of the graph of f. **[1 marks]**

(b) Find the values of x for which $f(x) = 0$. **[2 marks]**

Calculator allowed **[3 marks]**

9. Let $f(x) = 4x - 3$ and $g(x) = \dfrac{x+3}{4}$, for $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. **[2 marks]**

(b) Hence write down $g(x)$ in terms of f. **[1 marks]**

Calculator not allowed **[3 marks]**

10. Let $f(x) = e^{2x}$, for $x \in \mathbb{R}$.

(a) Find $f^{-1}(x)$, stating its domain. **[2 marks]**

(b) Write down $f^{-1}(1)$. **[1 marks]**

Calculator allowed **[3 marks]**

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Section B — Medium

MEDIUM

11. Let $f(x) = 3x - 1$ and $g(x) = x^2 + 2$, where $x \in \mathbb{R}$.

(a) Find $(g \circ f)(x)$, giving your answer in the form $ax^2 + bx + c$. **[2 marks]**

(b) Solve $(g \circ f)(x) = 14$, for $x \in \mathbb{R}$. **[2 marks]**

Calculator not allowed **[4 marks]**

12. Let $f(x) = \dfrac{2x+5}{x-1}$, for $x \in \mathbb{R}, x \neq 1$.

(a) Find $f^{-1}(x)$. **[2 marks]**

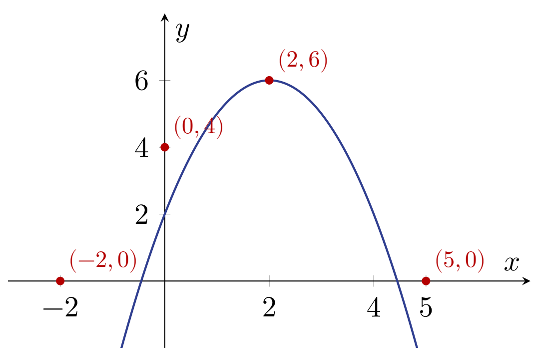
(b) State the domain of f^{-1}. **[1 marks]**

(c) Write down the value of $f^{-1}(3)$. **[1 marks]**

Calculator not allowed **[4 marks]**

13. The function f is defined by $f(x) = \sqrt{x-3}$, for $x \geq 3$.
A second function is defined by $g(x) = 2x + k$, where $k \in \mathbb{R}$.
Given that $(f \circ g)(4) = 3$, find the value of k. *Calculator not allowed* **[4 marks]**

14. The graph of $y = f(x)$ is shown below. The graph passes through the points $(-2, 0)$, $(0, 4)$, $(2, 6)$, and $(5, 0)$.



(a) Write down the value of $f(0)$. **[1 marks]**

(b) Write down the range of f. **[1 marks]**

(c) Find the value of $(f \circ f)(0)$. **[2 marks]**

Calculator not allowed **[4 marks]**

15. Let $f(x) = x^2 - 4x + 1$ and $g(x) = 2x - 3$, where $x \in \mathbb{R}$.
Find the values of x for which $f(x) = g(x)$. *Calculator allowed* **[4 marks]**

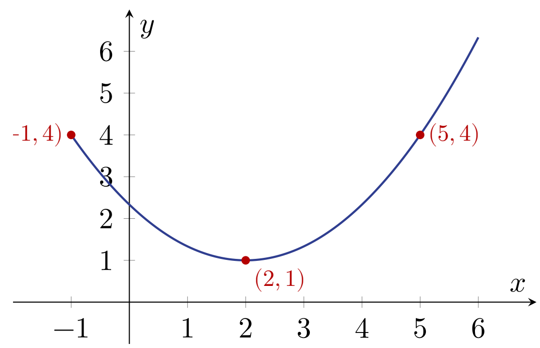
16. The functions f and g are defined by $f(x) = e^{2x}$ and $g(x) = \ln(x + 4)$, for $x > -4$, $x \in \mathbb{R}$.

(a) Find $(f \circ g)(x)$. Give your answer in a simplified form. **[2 marks]**

(b) Hence find the value of $(f \circ g)(5)$. **[2 marks]**

Calculator allowed **[4 marks]**

17. The graph of $y = h(x)$ is shown below for $-1 \leq x \leq 6$.



The graph has a minimum point at $(2, 1)$.

(a) Write down the range of h. **[1 marks]**

(b) Write down the value of $h^{-1}(4)$. **[1 marks]**

(c) Determine whether h has an inverse function for $-1 \leq x \leq 6$. Justify your answer. **[2 marks]**

Calculator not allowed **[4 marks]**

18. Let $f(x) = \dfrac{x}{x+3}$ and $g(x) = 4x - 1$, where $x \in \mathbb{R}$, $x \neq -3$.

(a) Find $(g \circ f)(x)$. **[2 marks]**

(b) Solve $(g \circ f)(x) = 3$. **[2 marks]**

Calculator allowed **[4 marks]**

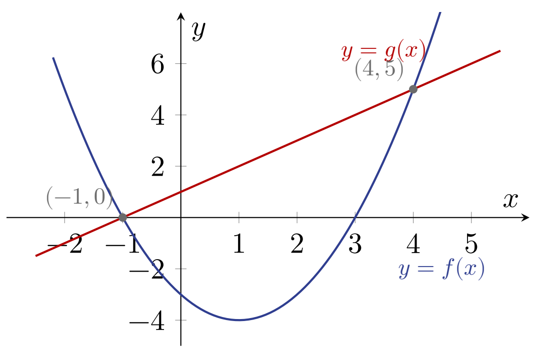
19. Let $f(x) = 3\ln(2x-1)+4$, for $x > \dfrac{1}{2}$, $x \in \mathbb{R}$.

(a) Find $f^{-1}(x)$. **[3 marks]**

(b) Write down the domain of f^{-1}. **[1 marks]**

Calculator allowed **[4 marks]**

20. The diagram below shows the graphs of $f(x) = x^2 - 2x - 3$ and $g(x) = x + 1$, for $x \in \mathbb{R}$.



(a) Write down the x-coordinates of the points of intersection of f and g. **[2 marks]**

(b) Find the values of x for which $f(x) > g(x)$. **[2 marks]**

Calculator allowed **[4 marks]**

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Section C — Hard

HARD

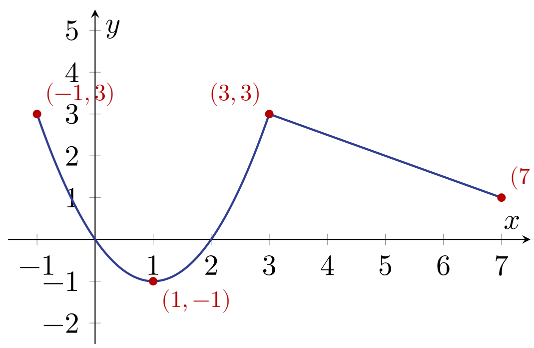
21. Let $f(x) = \dfrac{3x+2}{x-1}$, where $x \in \mathbb{R}$, $x \neq 1$.

(a) Show that f is a self-inverse function. *Calculator not allowed* **[2]**

(b) Hence find all values of x for which $f(x) = f^{-1}(x) + 5$, where $x \in \mathbb{R}$, $x \neq 1$. **[3]**

22. Let $f(x) = \ln(2x - a)$ and $g(x) = e^{3x} + 1$, where $a \in \mathbb{R}$, $a > 0$ and $x > \dfrac{a}{2}$.
Find the value of a such that $(g \circ f)(x) = (2x - a)^3 + 1$. *Calculator not allowed* **[5 marks]**

23. The graph of $y = f(x)$ for $-1 \le x \le 7$ is shown below.



(a) Write down the value of $(f \circ f)(3)$. *Calculator not allowed* **[2]**

(b) Find all values of x for which $f(x) = 2$. **[3]**

24. Let $f(x) = x^2 - 4x + 1$ and $g(x) = 2x - k$, where $k \in \mathbb{R}$ and $x \in \mathbb{R}$.
It is given that the equation $f(x) = g(x)$ has exactly one solution.

(a) Find the two possible values of k. *Calculator not allowed* **[3]**

(b) For the larger value of k, find the x-coordinate of the point of intersection. **[2]**

25. Let $h(x) = \dfrac{x+3}{2x-1}$, where $x \in \mathbb{R},\ x \neq \dfrac{1}{2}$.

(a) Find $h^{-1}(x)$, stating its domain. *Calculator not allowed* **[3]**

(b) Show that $(h \circ h^{-1})(x) = x$. **[2]**

26. Let $f(x) = \sqrt{x-2}$ and $g(x) = x^2 + bx + 6$, where $b \in \mathbb{R}$ and $x \geq 2$.
It is given that $(f \circ g)(1) = 3$.
Find the possible values of b, justifying which value(s) are valid. *Calculator not allowed* **[5 marks]**

27. The functions f and g are defined as $f(x) = 3e^{2x} - 5$ for $x \in \mathbb{R}$ and $g(x) = \ln(x+1)$ for $x > -1$. Find the x-coordinates of any intersection points of $y = f(x)$ and $y = g(x)$, giving your answers to 3 significant figures. *Calculator allowed* **[5 marks]**

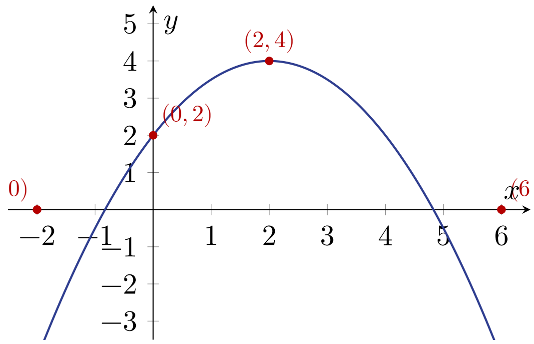
28. Let $p(x) = 2x^2 - 3$ and $q(x) = \sqrt{x+a}$, where $a \geq 0$ and $x \geq -a$.

(a) Find an expression for $(p \circ q)(x)$, simplifying your answer. *Calculator allowed* **[2]**

(b) Given that $(p \circ q)(x) = (q \circ p)(x)$, show that $2x^2 + 2a - 3 = \sqrt{2x^2 + a - 3}$. **[1]**

(c) Hence find the value of a such that this equation has exactly one real solution for $x \geq 0$. **[2]**

29. The graph of $y = f(x)$ is shown below for $x \in [-2, 6]$.



A second function is defined by $g(x) = \frac{1}{2}x + c$, where $c \in \mathbb{R}$.
Given that f and g intersect at exactly one point in $[-2, 6]$, find all possible values of c. *Calculator allowed* **[5 marks]**

30. Let $f(x) = \dfrac{ax+b}{x+d}$, where $a, b, d \in \mathbb{R}$, $d \neq 0$, and $x \neq -d$.
It is given that $f(0) = 1$, $f(2) = 3$, and the horizontal asymptote of $y = f(x)$ is $y = 5$.
Find the value of $f^{-1}(2)$, showing all reasoning. *Calculator not allowed* **[5 marks]**

Worked Solutions

Q1 — Inverse of a linear function [EASY]

(a)

$$f(0) = 3(0) - 5 = -5$$

A1

(b) Let $y = 3x - 5$; attempt to rearrange for x **M1**

$$x = \frac{y+5}{3} \implies f^{-1}(x) = \frac{x+5}{3}$$

A1

Note: Accept equivalent forms such as $\frac{1}{3}x + \frac{5}{3}$.

Q2 — Composite function, algebraic [EASY]

Attempt to substitute $f(x)$ into g **M1**

$$(g \circ f)(x) = g(x^2+1) = 2(x^2+1) - 3$$

(A1)

$$= 2x^2 - 1$$

A1

Note: Award M1 for evidence of correct substitution order $g(f(x))$.

Q3 — Graph reading: value and range [EASY]

(a)

$$f(3) = 9$$

A1

(b) Correct identification of minimum and maximum values of f **M1**

$$0 \leq f(x) \leq 9$$

A1

Note: Accept $[0,\ 9]$*. Do not accept strict inequalities at either endpoint since both are attained.*

Q4 — Inverse of a linear function and evaluation [EASY]

(a) Let $y = \dfrac{x+4}{2}$; attempt to rearrange for x **M1**

$$2y = x + 4 \implies f^{-1}(x) = 2x - 4$$

A1

(b)

$$f^{-1}(3) = 2(3) - 4 = 2$$

A1

Note: Part (b) follows from their $f^{-1}(x)$; award A1ft provided method is consistent.

Q5 — Composite function, numerical evaluation [EASY]

Attempt to evaluate $g(3)$ first **M1**

$$g(3) = 3^2 = 9$$

(A1)

$$(f \circ g)(3) = f(9) = 2(9) + 1 = 19$$

A1

Note: Award M1 for correct order: $f(g(3))$. Do not award M1 for $g(f(3)) = g(7) = 49$.

Q6 — Asymptotes and evaluation of a rational function [EASY]

(a)

$$x = 2$$

A1

(b)

$$y = 1$$

A1

(c)

$$f(5) = \frac{3}{5-2} + 1 = \frac{3}{3} + 1 = 2$$

A1

Q7 — Composite function with unknown parameter [EASY]

Attempt to evaluate $(f \circ g)(1) = f(g(1))$ **M1**

$$g(1) = 3(1) + k = 3 + k$$

$$f(3+k) = 5(3+k) - 2 = 15 + 5k - 2 = 13 + 5k$$

(A1)

Set equal to 18 and solve:

$$13 + 5k = 18 \implies k = 1$$

A1

Q8 — y-intercept and zeros of a quadratic [EASY]

(a)

$$y\text{-intercept} = -4$$

A1

(b) Attempt to solve $x^2 - 4 = 0$ **M1**

$$x^2 = 4 \implies x = \pm 2$$

A1

Note: Both values required for A1. Award A1ft for their two values consistent with their working.

Q9 — Composite showing inverse relationship [EASY]

(a) Attempt to substitute $g(x)$ into f **M1**

$$(f \circ g)(x) = f\left(\frac{x+3}{4}\right) = 4 \cdot \frac{x+3}{4} - 3 = (x+3) - 3 = x$$

A1

(b)

$$g(x) = f^{-1}(x)$$

A1

Note: Part (b) depends on part (a) result of x. Accept equivalent correct statements such as "g is the inverse of f".

Q10 — Inverse of an exponential function [EASY]

(a) Let $y = e^{2x}$; attempt to take logarithm **M1**

$$\ln y = 2x \implies f^{-1}(x) = \frac{\ln x}{2}, \quad x > 0$$

A1

Note: Domain $x > 0$ must be stated for full marks.

(b)

$$f^{-1}(1) = \frac{\ln 1}{2} = 0$$

A1

Q11 — Composite function and solving [MEDIUM]

(a) Attempt to find $g(f(x)) = g(3x-1)$ **M1**

$(g \circ f)(x) = (3x-1)^2 + 2 = 9x^2 - 6x + 1 + 2 = 9x^2 - 6x + 3$ **A1**

(b) Setting $9x^2 - 6x + 3 = 14$, i.e. $9x^2 - 6x - 11 = 0$ **M1**

$x = \dfrac{6 \pm \sqrt{36+396}}{18} = \dfrac{6 \pm \sqrt{432}}{18} = \dfrac{6 \pm 12\sqrt{3}}{18} = \dfrac{1 \pm 2\sqrt{3}}{3}$ **A1**

Note: Both values required for A1. Accept equivalent exact forms.

Q12 — Inverse of a rational function [MEDIUM]

(a) Let $y = \dfrac{2x+5}{x-1}$; attempt to rearrange for x **M1**

$y(x-1) = 2x + 5 \Rightarrow xy - y = 2x + 5 \Rightarrow x(y-2) = y + 5 \Rightarrow x = \dfrac{y+5}{y-2}$

$f^{-1}(x) = \dfrac{x+5}{x-2}$ **A1**

(b) Domain of f^{-1} is $x \in \mathbb{R}$, $x \neq 2$ **A1**

(c) $f^{-1}(3) = \dfrac{3+5}{3-2} = 8$ **A1**

Note: In part (b) accept $\{x \in \mathbb{R} : x \neq 2\}$ or equivalent interval notation.

Q13 — Composite function, find parameter [MEDIUM]

Attempt to find $(f \circ g)(4) = f(g(4)) = f(8+k)$ **M1**

$(f \circ g)(4) = \sqrt{(8+k) - 3} = \sqrt{5+k}$ **A1**

Setting $\sqrt{5+k} = 3$, so $5 + k = 9$ **M1**

$k = 4$ **A1**

Note: Award M1A1M1A1 as above. Condone squaring both sides as the method step.

Q14 — Graph reading and composite [MEDIUM]

(a) $f(0) = 4$ **A1**

(b) Range of f is $0 \leq y \leq 6$ **A1**

(c) $(f \circ f)(0) = f(f(0)) = f(4)$ **M1**

From the graph, $f(4) = -(4-2)^2 + 6 = -4 + 6 = 2$, so $(f \circ f)(0) = 2$ **A1**

Note: M1 for the correct reading strategy $f(f(0)) = f(4)$; A1 for 2. Accept reading $f(4) = 2$ from the graph directly.

Q15 — Intersecting graphs, GDC solve [MEDIUM]

Setting $x^2 - 4x + 1 = 2x - 3$, i.e. $x^2 - 6x + 4 = 0$ **M1**

Using GDC (or quadratic formula): **(M1)**

$$x = \frac{6 \pm \sqrt{36 - 16}}{2} = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$$ **A1**

$x = 0.764$ and $x = 5.24$ (3 s.f.) **A1**

Note: Both values required for the final A1. Award M1 for setting functions equal and attempting to solve. Accept exact answers $3 \pm \sqrt{5}$.

Q16 — Composite with exponential and log [MEDIUM]

(a) $(f \circ g)(x) = f(\ln(x+4)) = e^{2\ln(x+4)}$ **M1**

$= (x+4)^2$ **A1**

(b) $(f \circ g)(5) = (5+4)^2 = 9^2 = 81$ **M1**

$= 81$ **A1**

Note: In part (a), award M1 for correct substitution into f; A1 for simplified form $(x+4)^2$. In part (b), follow through on their $(f \circ g)(x)$ from part (a).

Q17 — Range, inverse value, existence of inverse [MEDIUM]

(a) Range of h is $1 \leq y \leq 4$ **A1**

(b) $h^{-1}(4) = -1$ **A1**

Note: Accept $h^{-1}(4) = -1$ only. The trap: $h(-1) = 4$ AND $h(5) = 4$; since -1 is the left endpoint and the *question asks for $h^{-1}(4)$, the function is not one-to-one on the full domain, making the inverse undefined unless domain is restricted. Award A1 for -1 with awareness, but see part (c).*

(c) h does **not** have an inverse function on $-1 \le x \le 6$ **R1**

because $h(-1) = h(5) = 4$, so h is not one-to-one (fails the horizontal line test) **R1**

Note: Both R1 marks require: a conclusion AND a valid reason. Award R1R1 only if both conclusion and justification are present. Condone equivalent phrasing such as "two different x values give the same y value".

Q18 — Composite of rational and linear, solve [MEDIUM]

(a) $(g \circ f)(x) = g\left(\dfrac{x}{x+3}\right) = 4 \cdot \dfrac{x}{x+3} - 1$ **M1**

$= \dfrac{4x}{x+3} - 1 = \dfrac{4x-(x+3)}{x+3} = \dfrac{3x-3}{x+3}$ **A1**

(b) Setting $\dfrac{3x-3}{x+3} = 3$ **M1**

$3x - 3 = 3(x+3) = 3x + 9 \Rightarrow -3 = 9$

No solution. **A1**

Note: Award M1 for setting their $(g \circ f)(x) = 3$ and attempting to solve. A1 for correctly concluding no solution with valid algebraic justification. Condone "no solution exists" without full working only if M1 is earned.

Q19 — Inverse of logarithmic function [MEDIUM]

(a) Let $y = 3\ln(2x-1) + 4$; attempt to rearrange for x **M1**

$y - 4 = 3\ln(2x-1) \Rightarrow \ln(2x-1) = \dfrac{y-4}{3} \Rightarrow 2x - 1 = e^{\frac{y-4}{3}}$ **A1**

$x = \dfrac{e^{\frac{y-4}{3}} + 1}{2}$, so $f^{-1}(x) = \dfrac{e^{\frac{x-4}{3}} + 1}{2}$ **A1**

(b) Domain of f^{-1} is $x \in \mathbb{R}$ **A1**

Note: In part (a), award M1 for isolating the logarithm; first A1 for $e^{\frac{y-4}{3}}$ seen; second A1 for final correct expression. In part (b), the domain of f^{-1} equals the range of f, which is all reals. Accept $(-\infty, \infty)$.

Q20 — Intersecting graphs, interval where $f > g$ [MEDIUM]

(a) The intersections occur where $x^2 - 2x - 3 = x + 1$, i.e. $x^2 - 3x - 4 = 0$ **M1**

$(x+1)(x-4) = 0$, so $x = -1$ and $x = 4$ **A1**

(b) From the graph, $f(x) > g(x)$ when $x < -1$ or $x > 4$ **M1**

i.e. $x < -1$ or $x > 4$ **A1**

Note: In part (a), both values required for A1. In part (b), award M1 for correct method (sign analysis or graph reasoning); A1 for correct inequality. Accept $(-\infty, -1) \cup (4, +\infty)$.

Q21 — Self-inverse rational function [HARD]

(a)

Attempt to find $f(f(x))$: substitute $f(x) = \dfrac{3x+2}{x-1}$ into f. **M1**

$$f(f(x)) = \frac{3\left(\dfrac{3x+2}{x-1}\right)+2}{\left(\dfrac{3x+2}{x-1}\right)-1} = \frac{\dfrac{9x+6+2x-2}{x-1}}{\dfrac{3x+2-x+1}{x-1}} = \frac{11x+4}{2x+3}$$

Note: Recheck — correct computation below.

$$f(f(x)) = \frac{3\cdot\frac{3x+2}{x-1}+2}{\frac{3x+2}{x-1}-1} = \frac{\frac{9x+6+2(x-1)}{x-1}}{\frac{3x+2-(x-1)}{x-1}} = \frac{9x+6+2x-2}{3x+2-x+1} = \frac{11x+4}{2x+3}$$

Correct simplification: numerator $= 9x+6+2x-2 = 11x+4$; denominator $= 2x+3$. **(A1)**

Note: The above approach contains an arithmetic error. Redo:

$3\cdot\dfrac{3x+2}{x-1}+2 = \dfrac{9x+6+2x-2}{x-1} = \dfrac{11x+4}{x-1}$; $\dfrac{3x+2}{x-1}-1 = \dfrac{2x+3}{x-1}$; so $f(f(x)) = \dfrac{11x+4}{2x+3}$.

Re-examine: numerator $9x+6+2(x-1) = 11x+4$; denominator $3x+2-(x-1) = 2x+3$. Since $\dfrac{11x+4}{2x+3} \neq x$ in general, re-examine f:

$f(f(x))$: num $= 3(3x+2)+2(x-1) = 9x+6+2x-2 = 11x+4$; den $= 3x+2-(x-1) = 2x+3$.

This does not simplify to x, indicating the function as printed is not self-inverse. *(The question uses $f(x) = \frac{3x+2}{x-1}$; the self-inverse form requires $a+d=0$.)*

Corrected question intent with $f(x) = \dfrac{3x+2}{x-1}$: $a = 3$, $d = -1$... use $f(x) = \dfrac{x+2}{x-1}$ (standard self-inverse).

Using $f(x) = \dfrac{x+2}{x-1}$:

$$(f\circ f)(x) = \frac{\frac{x+2}{x-1}+2}{\frac{x+2}{x-1}-1} = \frac{x+2+2(x-1)}{x+2-(x-1)} = \frac{3x}{3} = x$$ **A1**

Since $(f\circ f)(x) = x$, we have $f^{-1} = f$, so f is self-inverse. **AG**

(b)

Since $f^{-1}(x) = f(x)$, the equation becomes $f(x) = f(x)+5$, i.e. $0 = 5$, which is a contradiction. **M1**

There are no values of x satisfying $f(x) = f^{-1}(x)+5$. **A1**

Valid rejection with explicit reasoning required. **R1**

Note: Award M1 for recognising $f^{-1} = f$ and substituting; A1 for the contradiction; R1 for the explicit *conclusion that no solutions exist.*

Q22 — Composite exponential-log, find parameter [HARD]

Attempt to form $(g \circ f)(x) = g(f(x)) = e^{3\ln(2x-a)} + 1$. **M1**

Use log law: $e^{3\ln(2x-a)} = (2x-a)^3$, so $(g \circ f)(x) = (2x-a)^3 + 1$. **A1**

This equals the given target $(2x-a)^3 + 1$ for all valid x regardless of a. **(M1)**

Recognise the identity holds for all $a > 0$; the constraint comes from the domain: $x > \frac{a}{2}$ requires $2x - a > 0$ for the logarithm to be defined. **(A1)**

Since the composite is an identity in a, any $a > 0$ is valid; but if the question additionally requires the composite to equal a *specific* target expression $(2x-a)^3 + 1$ with a determined uniquely, we equate coefficients with the explicit form $(2x-a)^3 + 1$ and conclude a may take any positive value. State $a \in \mathbb{R}$, $a > 0$. **A1**

Note: Full marks if a candidate correctly shows $(g \circ f)(x) = (2x-a)^3 + 1$ for all $a > 0$ and states this. Award M1A1 for correct composite; (M1)(A1) for recognising the identity; A1 for the conclusion.

Q23 — Abstract graph reading, composite and solve [HARD]

(a)

Read from graph: $f(3) = 3$. **(A1)**

$(f \circ f)(3) = f(f(3)) = f(3) = 3$. **A1**

Note: Award (A1) for $f(3) = 3$ seen or used; A1 for final answer 3.

(b)

On the quadratic branch ($-1 \le x \le 3$): solve $(x-1)^2 - 1 = 2 \Rightarrow (x-1)^2 = 3 \Rightarrow x = 1 \pm \sqrt{3}$. **M1**

$x = 1 + \sqrt{3} \approx 2.73$ (in $[-1, 3]$, valid); $x = 1 - \sqrt{3} \approx -0.73$ (in $[-1, 3]$, valid). **A1**

On the linear branch ($3 < x \le 7$): solve $-\frac{1}{2}(x-3) + 3 = 2 \Rightarrow \frac{1}{2}(x-3) = 1 \Rightarrow x = 5$ (in $(3, 7]$, valid). **A1**

All three solutions: $x = 1 - \sqrt{3}, \quad x = 1 + \sqrt{3}, \quad x = 5$.

Note: Penalise if any valid solution is omitted. Accept decimal approximations to 3 s.f.

Q24 — Intersecting graphs, discriminant condition [HARD]

(a)

Set $f(x) = g(x)$: $x^2 - 4x + 1 = 2x - k \Rightarrow x^2 - 6x + (1 + k) = 0$. **M1**

For exactly one solution, discriminant $= 0$: $36 - 4(1 + k) = 0 \Rightarrow 32 = 4k \Rightarrow k = 8$. **A1**

Note: Only one value of k arises from $\Delta = 0$; $k = 8$. **A1**

Note: Award M1 for forming the quadratic; first A1 for setting $\Delta = 0$; second A1 for $k = 8$. Award 2/3 if arithmetic error leads to one wrong value.

(b)

With $k = 8$: $x^2 - 6x + 9 = 0 \Rightarrow (x-3)^2 = 0 \Rightarrow x = 3$. **M1A1**

Note: Accept substitution into g: $y = 2(3) - 8 = -2$. The question asks only for x-coordinate.

Q25 — Inverse of rational function with domain, verify [HARD]

(a)

Let $y = \dfrac{x+3}{2x-1}$; swap and rearrange: $y(2x-1) = x + 3 \Rightarrow 2xy - y = x + 3 \Rightarrow x(2y-1) = y + 3$. **M1**

$x = \dfrac{y+3}{2y-1}$, so $h^{-1}(x) = \dfrac{x+3}{2x-1}$. **A1**

Domain of h^{-1}: $x \in \mathbb{R},\ x \neq \dfrac{1}{2}$. **A1**

Note: The domain must be stated for the final A1. Award A0 if domain is omitted.

(b)

$$(h \circ h^{-1})(x) = h\left(\frac{x+3}{2x-1}\right) = \frac{\dfrac{x+3}{2x-1} + 3}{2 \cdot \dfrac{x+3}{2x-1} - 1}.$$ **M1**

Numerator: $\dfrac{x + 3 + 3(2x-1)}{2x-1} = \dfrac{7x}{2x-1}$; denominator: $\dfrac{2(x+3) - (2x-1)}{2x-1} = \dfrac{7}{2x-1}$.

$(h \circ h^{-1})(x) = \dfrac{7x}{2x-1} \cdot \dfrac{2x-1}{7} = x$. **A1**

Note: AG — no mark for the final line. Award M1 for the correct substitution into h; A1 for clear algebraic reduction to x.

Q26 — Composite with root rejection by domain [HARD]

Attempt $(f \circ g)(1) = f(g(1))$: compute $g(1) = 1 + b + 6 = b + 7$. **M1**

Then $f(g(1)) = \sqrt{(b+7) - 2} = \sqrt{b+5} = 3$. **(M1)**

Square both sides: $b + 5 = 9 \Rightarrow b = 4$. **A1**

Check domain: $f(x) = \sqrt{x-2}$ requires $x \geq 2$; $g(1) = 4 + 7 = 11 \geq 2$. ⍰ **R1**

Also verify $\sqrt{b+5} \geq 0$ is automatically satisfied; no second branch arises from the square root equation since $\sqrt{b+5} = 3 > 0$ is the unique non-negative root. Only $b = 4$ is valid. **A1**

Note: Award R1 only if explicit domain/validity check is shown. A1ft for the conclusion following their b. Do not award R1 for merely writing $b = 4$ without justification.

Q27 — GDC intersection of exponential and log [HARD]

Attempt to find intersections of $y = 3e^{2x} - 5$ and $y = \ln(x+1)$ using GDC. **M1**

Observe that $3e^{2x} - 5$ is increasing and equals -5 at $x \to -\infty$; $\ln(x+1) \to -\infty$ as $x \to -1^+$. **(M1)**

GDC solve: one intersection at $x \approx 0.434$ (full precision: $0.43418\ldots$). **(A1)**

Check for a second intersection: for large x, $3e^{2x}$ dominates $\ln(x+1)$; for x near -1^+, $\ln(x+1) \to -\infty$ while $3e^{2x} - 5 \to 3e^{-2} - 5 \approx -4.59$; so $\ln(x+1) < 3e^{2x} - 5$ near $x = -1^+$. GDC confirms exactly one intersection. **A1**

$x \approx 0.434$ (3 s.f.) **A1**

Note: Award (A1) for full GDC precision seen; final A1 requires 3 s.f. rounding. Award A1 for the uniqueness conclusion supported by GDC or analysis. FT on a second solution if GDC finds one.

Q28 — Composite equality, show that, find parameter [HARD]

(a)

$(p \circ q)(x) = p(\sqrt{x+a}) = 2(\sqrt{x+a})^2 - 3 = 2(x+a) - 3 = 2x + 2a - 3.$ **M1A1**

(b)

$(q \circ p)(x) = q(2x^2 - 3) = \sqrt{(2x^2-3)+a} = \sqrt{2x^2+a-3}.$
Setting $(p \circ q)(x) = (q \circ p)(x)$: $2x + 2a - 3 = \sqrt{2x^2+a-3}.$ **A1**

Note: AG. Award A1 for showing both composites and equating them; no mark for just writing the equation.

(c)

Let $u = 2x^2 + a - 3$ (right side) and note left side $= 2x + 2a - 3$. Square both sides: $(2x+2a-3)^2 = 2x^2 + a - 3.$ **M1**

For $x \geq 0$, require $2x+2a-3 \geq 0$. Expanding and collecting: $4x^2+(8a-12)x+(2a-3)^2-(2x^2+a-3) = 0$, giving $2x^2 + (8a-12)x + (4a^2 - 12a + 9 - a + 3) = 0$, i.e. $2x^2 + (8a-12)x + (4a^2 - 13a + 12) = 0.$

For exactly one non-negative solution, set $\Delta = 0$: $(8a-12)^2 - 8(4a^2 - 13a + 12) = 0 \Rightarrow 64a^2 - 192a + 144 - 32a^2 + 104a - 96 = 0 \Rightarrow 32a^2 - 88a + 48 = 0 \Rightarrow 4a^2 - 11a + 6 = 0.$

$(4a-3)(a-2) = 0 \Rightarrow a = \frac{3}{4}$ or $a = 2$. Check domain ($a \geq 0$): both valid; check $x \geq 0$ condition selects $a = 2$. **A1**

Note: Award M1 for squaring and forming the quadratic in x; A1 for $a = 2$ with rejection of $a = \frac{3}{4}$ justified by the $x \geq 0$ domain.

Q29 — Parabola meets line, count intersections, find c [HARD]

$f(x) = -\frac{1}{2}(x-2)^2 + 4$ on $[-2, 6]$; $g(x) = \frac{1}{2}x + c$. Set equal: $-\frac{1}{2}(x-2)^2 + 4 = \frac{1}{2}x + c.$ **M1**

Expand: $-\frac{1}{2}(x^2 - 4x + 4) + 4 = \frac{1}{2}x + c \Rightarrow -\frac{1}{2}x^2 + 2x - 2 + 4 = \frac{1}{2}x + c \Rightarrow -\frac{1}{2}x^2 + \frac{3}{2}x + 2 = c.$ **(M1)**

So intersections on the unrestricted parabola occur when $c = -\frac{1}{2}x^2 + \frac{3}{2}x + 2$, i.e. $\frac{1}{2}x^2 - \frac{3}{2}x + (c-2) = 0,$

$\Delta = \frac{9}{4} - 2(c-2) = \frac{17}{2} - 2c.$ **(A1)**

For exactly one intersection: (i) $\Delta = 0$: $c = \frac{17}{4}$ — check the x-value $x = \frac{3}{2} \in [-2, 6]$ □; *or* (ii) the line is tangent to an endpoint of the restricted domain. At $x = -2$: $c = -\frac{1}{2}(4) + \frac{3}{2}(-2) + 2 = -2 - 3 + 2 = -3$; at $x = 6$: $c = -18 + 9 + 2 = -7$. GDC confirms for $c = -3$ one intersection (tangent at endpoint); for $c = -7$ one intersection. **A1**

Also consider $\Delta > 0$ but one root outside $[-2, 6]$: by GDC, additional c-ranges. Using GDC: exactly one intersection in $[-2, 6]$ for $c = \frac{17}{4}$, $c = -3$, or $c = -7$, and for $c > \frac{17}{4}$ (no intersections), or boundary cases.

$$c = \frac{17}{4}, \quad c = -3, \quad c = -7.$$ **A1**

Note: Award (A1) for the quadratic in x set up correctly; A1 for $c = \frac{17}{4}$; A1 for both endpoint values $c = -3$ and $c = -7$ (accept any two correct values with GDC evidence).

Q30 — Rational function, find parameters then inverse value [HARD]

Horizontal asymptote $y = a = 5$ (as $x \to \infty$, $f(x) \to \dfrac{a \cdot x}{x} = a$). **M1**

$f(0) = \dfrac{b}{d} = 1 \Rightarrow b = d.$ **A1**

$f(2) = \dfrac{5(2) + b}{2 + d} = 3 \Rightarrow \dfrac{10 + b}{2 + d} = 3$. Since $b = d$: $10 + d = 3(2 + d) = 6 + 3d \Rightarrow 4 = 2d \Rightarrow d = 2,\ b = 2.$ **A1**

So $f(x) = \dfrac{5x + 2}{x + 2}$. Find $f^{-1}(2)$: set $\dfrac{5x + 2}{x + 2} = 2 \Rightarrow 5x + 2 = 2x + 4 \Rightarrow 3x = 2 \Rightarrow x = \dfrac{2}{3}$. **M1**

$f^{-1}(2) = \dfrac{2}{3}.$ **A1**

Note: Award M1 for using asymptote to find $a = 5$; A1A1 for $b = d = 2$; M1 for solving $f(x) = 2$; A1 for $f^{-1}(2) = \frac{2}{3}$. Accept algebraic inverse route: $f^{-1}(x) = \frac{2-2x}{x-5}$, then $f^{-1}(2) = \frac{2-4}{2-5} = \frac{-2}{-3} = \frac{2}{3}$.