IB Math AA SL — Topic 4
Probability Distributions

Discrete Probability Distributions · Expected Values

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:_____ Date:_____

Score:_____/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- Sum of all probabilities: $\sum_{x} P(X = x) = 1$
- Expected value: $E(X) = \sum_{x} x \cdot P(X = x)$
- Expected value of a function: $E(g(X)) = \sum_{x} g(x) \cdot P(X = x)$
- Conditional probability: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
- Expected gain in a game (stake s): $E(\text{gain}) = E(X) - s$; fair game when $E(\text{gain}) = 0$
- Complementary probability: $P(X \leq k) = 1 - P(X > k)$
- Fair ticket price: set $E(\text{winnings}) = \text{price}$, so price $= \sum_{x} x \cdot P(X = x)$

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GDC Tips

- **Storing a table:** Enter x-values in List 1 and $P(X = x)$ values in List 2; verify the sum of List 2 equals 1 using `sum(L2)`.
- **Computing $E(X)$:** Use `sum(L1*L2)` after entering values in List 1 and List 2; this directly evaluates $\sum_x x \cdot P(X = x)$.
- **Solving for a parameter k:** Use `solve()` or the equation solver with the sum-to-1 constraint typed explicitly; check the result satisfies all probability conditions ($0 \leq P \leq 1$).
- **Computing $E(g(X))$:** Define List 3 $= g$(List 1), then evaluate `sum(L3*L2)` for quantities such as $E(X^2)$ or $E(2X + 1)$.
- **Conditional probabilities:** Store numerator and denominator events as separate sums from the table; divide them directly rather than retyping fractions.
- **Two-unknown systems:** Enter both the sum-to-1 equation and the $E(X)$ equation into `solve({...,...},{a,k})` simultaneously to avoid sign errors from manual substitution.

Common Mistakes to Avoid

1. **Forgetting to verify sum-to-1.** After finding a parameter k, students substitute back to check $E(X)$ but neglect to confirm every individual $P(X = x) \geq 0$ and that the total is exactly 1. Any value of k that makes a probability negative or greater than 1 must be rejected even if the algebra is correct.
2. **Confusing $E(X)$ with the fair ticket price.** The fair price equals $E(\text{winnings})$, not $E(\text{net gain})$. Students who subtract the stake before equating to zero arrive at a price that is off by the stake amount. Always separate gross winnings from the cost of play before equating.
3. **Incorrect conditional probability numerator.** For $P(X > s \mid X < t)$, students frequently use $P(X > s)$ in the numerator instead of $P(s < X < t)$. The numerator must be the intersection, read directly from the table rows satisfying both inequalities simultaneously.
4. **Using $E(X^2)$ instead of $[E(X)]^2$.** Variance is not in AA SL, but questions on $E(X^2)$ or $E(2X - 3)$ do appear. Students mis-square the expected value rather than computing the expectation of the squared variable; these two quantities are generally not equal.
5. **Wrong direction in a two-unknown simultaneous system.** When two equations (sum-to-1 and a given $E(X)$) are set up, students sometimes write the $E(X)$ equation using the P-values instead of the x-values as multipliers, or vice versa. Write both equations in full before solving; do not compress steps.

6. **Omitting the parameter order condition.** When a question states that one parameter is less than another, both solutions of a symmetric system satisfy the equations algebraically, but only one respects the ordering constraint. Failing to check and declare which solution is accepted loses the final accuracy mark, even when the method is otherwise perfect.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

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1. EASY No Calculator [3 marks]

The discrete random variable X has the following probability distribution, where $k \in \mathbb{R}$.

x	1	2	3	4
$P(X = x)$	0.1	k	0.3	$2k$

Find the value of k.

2. EASY Calculator [3 marks]

The discrete random variable X has the following probability distribution.

x	0	1	2	3	4
$P(X = x)$	0.05	0.20	0.35	0.25	0.15

Find $E(X)$.

3. EASY Calculator [3 marks]

A discrete random variable X takes values 1, 2, and 3 with probabilities 0.4, 0.35, and 0.25 respectively. Find $E(X)$.

4. **EASY** **No Calculator** **[3 marks]**

The discrete random variable X has the following probability distribution, where $a \in \mathbb{R}$.

x	2	4	6
$P(X = x)$	a	$3a$	a

Find the value of a.

5. **EASY** **Calculator** **[3 marks]**

A discrete random variable X takes values 0, 1, 2, and 3 with $P(X = 0) = 0.3$, $P(X = 1) = 0.4$, $P(X = 2) = 0.2$, and $P(X = 3) = 0.1$. Find $P(X \leq 1)$.

6. **EASY** **No Calculator** **[3 marks]**

The discrete random variable X has the following probability distribution, where $k \in \mathbb{R}$.

x	1	2	3	4
$P(X = x)$	$2k$	k	$3k$	k

Find the value of k, and hence find $E(X)$.

7. **EASY** **Calculator** **[3 marks]**

A spinner has four sectors numbered 1, 2, 3, and 4. The probability of landing on each sector is equal. Let X be the number shown after one spin, where $X \in \{1, 2, 3, 4\}$. Find $E(X)$.

8. EASY No Calculator **[3 marks]**

A discrete random variable X takes values 0, 5, and 10 with probabilities $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$ respectively. Find $E(X)$.

9. EASY Calculator **[3 marks]**

A game gives a player a prize of \$3 with probability 0.2, a prize of \$1 with probability 0.5, and no prize with probability 0.3. Let X be the prize amount in dollars received by a player. Find $E(X)$.

10. EASY Calculator **[3 marks]**

A discrete random variable X takes values 1, 2, 3, 4, and 5, each with equal probability. Find $E(X)$.

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator [4 marks]

The discrete random variable X has the following probability distribution, where $k \in \mathbb{R}$.

x	1	2	3	4
$P(X = x)$	$2k$	k	$3k$	0.2

Find the value of k.

12. MEDIUM Calculator [4 marks]

The discrete random variable X has the following probability distribution, where $a, b \in \mathbb{R}$.

x	0	1	2	3
$P(X = x)$	0.1	a	b	0.15

Given that $E(X) = 1.7$, find the values of a and b.

13. MEDIUM No Calculator [4 marks]

A fair six-sided die has faces numbered 1, 1, 2, 3, 4, 5. The die is rolled once and the number showing, X, is recorded.

(a) Write down $P(X = 1)$. [1]

(b) Find $E(X)$. [3]

14. MEDIUM Calculator [4 marks]

The discrete random variable X has the following probability distribution, where $k \in \mathbb{R}$.

x	-1	0	1	2	3
$P(X = x)$	0.05	k	0.25	$3k$	0.10

(a) Find the value of k. [2]

(b) Find $E(X)$. [2]

15. MEDIUM Calculator [4 marks]

Amara plays a game in which she rolls a fair four-sided die with faces numbered 1, 2, 3, 4. She wins \$5 if the die shows a 4, wins \$2 if the die shows a 3, and loses \$1 otherwise. Let W be Amara's winnings, in dollars, on a single play of the game.

(a) Write down all possible values of W. [1]

(b) Find $E(W)$. [3]

16. MEDIUM No Calculator [4 marks]

The discrete random variable X has the following probability distribution, where $k \in \mathbb{R}^+$.

x	1	2	3
$P(X = x)$	k^2	$2k$	0.16

Find the value of k.

17. MEDIUM Calculator [4 marks]

Beto records the number of text messages, X, he receives each morning before 9 am. The probability distribution of X is given below, where $a \in \mathbb{R}$.

x	0	1	2	3
$P(X = x)$	0.3	0.4	a	0.05

(a) Find the value of a. [1]

(b) Find $P(X \leq 2 \mid X \geq 1)$. [3]

18. MEDIUM Calculator [4 marks]

A charity raffle sells 200 tickets at \$3 each. There is one prize of \$150 and two prizes of \$50. Chiara buys one ticket. Let G be Chiara's net gain, in dollars, from buying the ticket.

(a) Write down the three possible values of G. [1]

(b) Find $E(G)$. [3]

19. MEDIUM No Calculator **[4 marks]**

The discrete random variable X has probability distribution given by

$$P(X = x) = \frac{c}{x}, \quad x \in \{1, 2, 4, 5\},$$

where $c \in \mathbb{R}^+$.

(a) Find the exact value of c. [2]

(b) Find $E(X)$, giving your answer as an exact fraction. [2]

20. MEDIUM Calculator **[4 marks]**

Dina plays a game by spinning a fair spinner with sectors labelled 1, 2, 3, 4, 5, 6. She pays $\$t$ to play. She receives \$10 if the spinner lands on 6, \$4 if it lands on 5, and nothing otherwise.

(a) Find $E(W)$, the expected winnings (before subtracting the cost), in dollars. [2]

(b) Find the value of t for which the game is fair. [2]

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD No Calculator [5 marks]

The discrete random variable X has the following probability distribution, where $k \in \mathbb{R}$, $k > 0$.

x	-2	-1	0	1	2
$P(X = x)$	k^2	$2k$	$3k$	$2k$	k^2

Show that $k = \dfrac{-7 + \sqrt{57}}{4}$.

22. HARD Calculator [5 marks]

A discrete random variable X takes values 1, 2, 3, 4 with probabilities as shown in the table below, where $a, b \in \mathbb{R}$, $a > 0, b > 0$.

x	1	2	3	4
$P(X = x)$	a	b	$2a$	$3b$

Given that $E(X) = 2.9$, find the values of a and b, and hence find $P(X \geq 3)$.

23. HARD Calculator [5 marks]

Sana plays a game in which she rolls a fair six-sided die. If the result is odd, she wins triple that number of dollars. If the result is even, she loses that number of dollars. Find the expected gain per game, and hence determine the minimum whole-number ticket price (in dollars) that makes the game unfavourable for the player.

24. HARD No Calculator [5 marks]

The discrete random variable X has $P(X = x) = c(4 - x^2)$ for $x \in \{-1, 0, 1\}$, where $c \in \mathbb{R}$, $c > 0$.

(a) Find the value of c. [2]

(b) Find $P(X = 1 \mid X \geq 0)$. [3]

25. HARD Calculator [5 marks]

Yuna and Dmitri each independently select a single integer at random from the set $\{1, 2, 3, 4, 5\}$. Let X be the absolute difference between their two chosen numbers. Find $E(X)$.

26. **HARD** **No Calculator** **[5 marks]**

A biased coin has probability p of showing heads, where $0 < p < 1$. The coin is tossed repeatedly until a head appears. Let X be the number of tosses required. It is given that $P(X = 1) = 3P(X = 3)$. Find the value of p, clearly rejecting any values that are not valid.

27. **HARD** **No Calculator** **[5 marks]**

The discrete random variable X takes values in $\{0, 1, 2, 3\}$. The probabilities satisfy $P(X = 0) = P(X = 1) = a$ and $P(X = 2) = P(X = 3) = b$, where $a, b > 0$. Given that $E(X) = \frac{3}{2}$, show that $a = b = \frac{1}{4}$.

28. **HARD** **Calculator** **[5 marks]**

Fatima plays a lottery game. She pays \$5 to enter. A single integer is selected uniformly at random from $\{1, 2, 3, \ldots, 20\}$. Fatima wins \$50 if the number is a perfect square, \$10 if the number is even but not a perfect square, and nothing otherwise. Let G be Fatima's net gain in dollars from one game. Find $E(G)$, and determine whether the game is fair.

29. **HARD** **Calculator** **[5 marks]**

The discrete random variable X has the probability distribution shown, where $k \in \mathbb{R}$, $k > 0$.

x	0	1	2	3	4
$P(X = x)$	0.1	k	0.3	$2k$	0.1

(a) Find the value of k. [2]

(b) Find $P(1 \leq X \leq 3 \mid X > 0)$. [3]

30. **HARD** **Calculator** **[5 marks]**

The random variable X is defined as follows: a fair four-sided die (faces 1, 2, 3, 4) is rolled. If the result is greater than 1, that value is recorded as X. If the result is 1, the die is rolled a second time and the sum of both rolls is recorded as X. Find $E(X)$.

Mark Schemes

Q1 — Sum-to-one find k EASY

Probabilities sum to 1:

$$0.1 + k + 0.3 + 2k = 1$$

M1

$3k = 0.6 \implies k = 0.2$ **A1**

Note: Award M1 for a correct sum-to-one equation with all four terms present. A1 for $k = 0.2$ only. Condone $k = \frac{1}{5}$.

Q2 — Expected value from a table EASY

$E(X) = 0(0.05) + 1(0.20) + 2(0.35) + 3(0.25) + 4(0.15)$ **M1**
$= 0 + 0.20 + 0.70 + 0.75 + 0.60$ **(A1)**
$E(X) = 2.25$ **A1**

Note: Award M1 for an attempt to evaluate $\sum x \cdot P(X = x)$ with at least three correct products. Final A1 for 2.25.

Q3 — Expected value from three outcomes EASY

$E(X) = 1(0.4) + 2(0.35) + 3(0.25)$ **M1**
$= 0.4 + 0.70 + 0.75$ **(A1)**
$E(X) = 1.85$ **A1**

Note: Award M1 for a correct substitution into $\sum x \cdot P(X = x)$. Final A1 for 1.85.

Q4 — Sum-to-one find a EASY

Probabilities sum to 1:

$$a + 3a + a = 1$$

M1

$5a = 1 \implies a = \dfrac{1}{5}$ **A1**

Note: Award M1 for a correct sum-to-one equation. A1 for $a = \frac{1}{5}$ or 0.2. Accept either exact form.

Q5 — Cumulative probability EASY

$P(X \leq 1) = P(X = 0) + P(X = 1)$ **M1**

$= 0.3 + 0.4$ **(A1)**

$P(X \leq 1) = 0.7$ **A1**

Note: Award M1 for identifying and summing the correct two probabilities. Final A1 for 0.7.

Q6 — Sum-to-one then expected value EASY

$2k + k + 3k + k = 1 \implies 7k = 1 \implies k = \dfrac{1}{7}$ **M1**

$E(X) = 1\left(\dfrac{2}{7}\right) + 2\left(\dfrac{1}{7}\right) + 3\left(\dfrac{3}{7}\right) + 4\left(\dfrac{1}{7}\right) = \dfrac{2+2+9+4}{7} = \dfrac{17}{7}$ **A1**

$E(X) = \dfrac{17}{7}$ **A1**

Note: First A1 for $k = \frac{1}{7}$. Second A1 for $E(X) = \frac{17}{7}$. Accept $2\frac{3}{7}$ or exact equivalent. Condone 2.43 (3 s.f.) only if exact form seen.

Q7 — Uniform discrete expected value EASY

Each value has probability $\dfrac{1}{4}$.

$E(X) = 1\left(\dfrac{1}{4}\right) + 2\left(\dfrac{1}{4}\right) + 3\left(\dfrac{1}{4}\right) + 4\left(\dfrac{1}{4}\right)$ **M1**

$= \dfrac{1+2+3+4}{4} = \dfrac{10}{4}$ **(A1)**

$E(X) = 2.5$ **A1**

Note: Award M1 for correct substitution. Accept $\frac{5}{2}$ or 2.5.

Q8 — Expected value with exact fractions EASY

$E(X) = 0\left(\dfrac{1}{2}\right) + 5\left(\dfrac{1}{3}\right) + 10\left(\dfrac{1}{6}\right)$ **M1**

$= 0 + \dfrac{5}{3} + \dfrac{10}{6} = \dfrac{10}{6} + \dfrac{10}{6}$ **(A1)**

$E(X) = \dfrac{10}{3}$ **A1**

Note: Award M1 for correct substitution into $\sum x \cdot P(X = x)$. Final A1 for $\frac{10}{3}$. Condone $3\frac{1}{3}$. Do not accept a decimal approximation without the exact form.

Q9 — Expected prize value EASY

$E(X) = 3(0.2) + 1(0.5) + 0(0.3)$ **M1**
$= 0.6 + 0.5 + 0$ **(A1)**
$E(X) = \$1.10$ **A1**

Note: Award M1 for an attempt at $\sum x \cdot P(X = x)$ with at least two correct products. Final A1 for 1.10. Accept 1.1.

Q10 — Uniform discrete expected value five values EASY

Each value has probability $\frac{1}{5}$.

$E(X) = (1 + 2 + 3 + 4 + 5) \times \frac{1}{5}$ **M1**
$= \frac{15}{5}$ **(A1)**
$E(X) = 3$ **A1**

Note: Award M1 for correct substitution into $\sum x \cdot P(X = x)$. Accept any correct grouping. Final A1 for 3.

Q11 — Find k from a probability table MEDIUM

Probabilities sum to 1: **M1**

$$2k + k + 3k + 0.2 = 1$$

$$6k = 0.8$$

$$k = \frac{0.8}{6} = \frac{2}{15} \approx 0.133$$

A1

Note: Accept $k = \frac{2}{15}$ or any exact equivalent; condone 0.133 (3 s.f.) if the exact form is seen.

Q12 — Two unknowns from sum-to-one and expected value MEDIUM

Sum to 1: $0.1 + a + b + 0.15 = 1 \implies a + b = 0.75$ **M1**
Expected value equation: $0(0.1) + 1(a) + 2(b) + 3(0.15) = 1.7 \implies a + 2b = 1.25$ **M1**
Subtracting: $b = 0.5$, hence $a = 0.25$ **A1A1**

Note: Award A1 for each correct value. Accept answers obtained by GDC solver with evidence.

Q13 — Expected value of a biased die MEDIUM

(a) $P(X=1)=\frac{2}{6}=\frac{1}{3}$ **A1**

(b) The full distribution is $P(X=1)=\frac{2}{6}$, $P(X=2)=\frac{1}{6}$, $P(X=3)=\frac{1}{6}$, $P(X=4)=\frac{1}{6}$, $P(X=5)=\frac{1}{6}$ **(M1)**

$E(X)=1\cdot\frac{2}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}$ **M1**

$=\frac{2+2+3+4+5}{6}=\frac{16}{6}=\frac{8}{3}$ **A1**

Note: Accept 2.67 (3 s.f.) alongside $\frac{8}{3}$. Do not accept 2.67 alone.

Q14 — Find k then expected value MEDIUM

(a) Sum to 1: $0.05+k+0.25+3k+0.10=1 \implies 4k=0.60 \implies k=0.15$ **M1A1**

(b) Using $k=0.15$, so $P(X=0)=0.15$, $P(X=2)=0.45$: **(M1)**

$E(X)=(-1)(0.05)+0(0.15)+1(0.25)+2(0.45)+3(0.10)$

$=-0.05+0+0.25+0.90+0.30=1.40$ **A1**

Note: Follow through their k in part (b) provided it yields a valid distribution.

Q15 — Expected winnings from a game MEDIUM

(a) $W\in\{-1,2,5\}$ **A1**

(b) Probabilities: $P(W=-1)=\frac{2}{4}=0.5$, $P(W=2)=\frac{1}{4}=0.25$, $P(W=5)=\frac{1}{4}=0.25$ **M1**

$E(W)=(-1)(0.5)+2(0.25)+5(0.25)$ **(M1)**

$=-0.5+0.5+1.25=1.25$ **A1**

Note: Accept \$1.25. Award M1 for correct probability assignment and (M1) for correct $E(W)$ structure.

Q16 — Quadratic equation for k from sum-to-one MEDIUM

Sum to 1:

$$k^2+2k+0.16=1 \implies k^2+2k-0.84=0$$

M1

Multiply by 25: $25k^2+50k-21=0$. Applying the quadratic formula:

$$k=\frac{-50\pm\sqrt{2500+2100}}{50}=\frac{-50\pm\sqrt{4600}}{50}=\frac{-50\pm10\sqrt{46}}{50}=-1\pm\frac{\sqrt{46}}{5}$$

M1

Since $k\in\mathbb{R}^+$, the negative root $k=-1-\frac{\sqrt{46}}{5}$ is rejected. **A1**

$$k = -1 + \frac{\sqrt{46}}{5} \approx 0.356 \text{ (3 s.f.)}$$

A1

Note: Award first M1 for forming the sum-to-one equation. Award second M1 for a valid method to solve the quadratic. Third A1 for explicit rejection of the negative root. Final A1 for $k = -1 + \frac{\sqrt{46}}{5} \approx 0.356$ only. Verification: $0.356^2 + 2(0.356) + 0.16 = 0.1267 + 0.712 + 0.16 = 0.9987 \approx 1$ (rounding), and exactly $\left(-1 + \frac{\sqrt{46}}{5}\right)^2 + 2\left(-1 + \frac{\sqrt{46}}{5}\right) + 0.16 = 1$ by construction.

Q17 — Find a and a conditional probability MEDIUM

(a) $0.3 + 0.4 + a + 0.05 = 1 \implies a = 0.25$ **A1**

(b)

$$P(X \leq 2 \mid X \geq 1) = \frac{P(1 \leq X \leq 2)}{P(X \geq 1)}$$

M1

Numerator: $P(X = 1) + P(X = 2) = 0.4 + 0.25 = 0.65$ **(M1)**
Denominator: $P(X \geq 1) = 0.4 + 0.25 + 0.05 = 0.70$

$$P(X \leq 2 \mid X \geq 1) = \frac{0.65}{0.70} = \frac{13}{14} \approx 0.929$$

A1

Note: Accept $\frac{13}{14}$ or 0.929 (3 s.f.). Follow through their a if $0 < a < 0.3$.

Q18 — Expected net gain from a raffle ticket MEDIUM

(a) Net gain values: $G \in \{147, 47, -3\}$ **A1**

Note: Award A1 for all three correct values. Winnings minus the $3 ticket cost.

(b)

$$P(G = 147) = \frac{1}{200}, \quad P(G = 47) = \frac{2}{200}, \quad P(G = -3) = \frac{197}{200}$$

M1

$$E(G) = 147 \cdot \frac{1}{200} + 47 \cdot \frac{2}{200} + (-3) \cdot \frac{197}{200}$$

(M1)

$$= \frac{147 + 94 - 591}{200} = \frac{-350}{200} = -1.75$$

A1

Note: Accept $-\$1.75$. Condone writing $E(G) = -1.75$ without the dollar sign.

Q19 — Exact c and expected value from a formula MEDIUM

(a) Sum to 1:

$$\frac{c}{1}+\frac{c}{2}+\frac{c}{4}+\frac{c}{5}=1$$

M1

$$c\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}\right)=1 \implies c\cdot\frac{20+10+5+4}{20}=1 \implies c\cdot\frac{39}{20}=1 \implies c=\frac{20}{39}$$

A1

(b)

$$E(X)=1\cdot\frac{20}{39}+2\cdot\frac{10}{39}+4\cdot\frac{5}{39}+5\cdot\frac{4}{39}$$

M1

$$=\frac{20+20+20+20}{39}=\frac{80}{39}$$

A1

Note: Award M1 for substituting their c into $\sum x\cdot P(X=x)$ with at least two correct terms.

Q20 — Expected winnings and a fair ticket price MEDIUM

(a)

$$E(W)=10\cdot\frac{1}{6}+4\cdot\frac{1}{6}+0\cdot\frac{4}{6}$$

M1

$$=\frac{10+4}{6}=\frac{14}{6}=\frac{7}{3}\approx 2.33\ (\$)$$

A1

(b) For a fair game, the cost equals the expected winnings: **M1**

$$t=\frac{7}{3}\approx 2.33$$

A1

Note: Accept $t=\frac{7}{3}$ or 2.33 (2 d.p.) since money context. Award M1 for setting $t=E(W)$ or equivalent fair-game condition. Follow through their $E(W)$ from part (a).

Q21 — Show that k solves a quadratic HARD

Sum of probabilities equals 1:

$$k^2+2k+3k+2k+k^2=1$$

M1

$$2k^2 + 7k - 1 = 0$$

A1

Applying the quadratic formula:

$$k = \frac{-7 \pm \sqrt{49+8}}{4} = \frac{-7 \pm \sqrt{57}}{4}$$

M1

Since $k > 0$, the solution $k = \dfrac{-7-\sqrt{57}}{4}$ is rejected. **R1**

Therefore $k = \dfrac{-7+\sqrt{57}}{4}$. **AG**

Note: Final AG line earns nothing; R1 requires the explicit reason that the negative root is rejected because $k > 0$.

Q22 — Two-unknown distribution with an expected value condition HARD

Sum to 1: $a + b + 2a + 3b = 1 \implies 3a + 4b = 1$ **M1**

Expected value equation: $1 \cdot a + 2 \cdot b + 3 \cdot 2a + 4 \cdot 3b = 2.9 \implies 7a + 14b = 2.9$ **A1**

Solving simultaneously: from $3a + 4b = 1$: $a = \dfrac{1-4b}{3}$

$$7 \cdot \frac{1-4b}{3} + 14b = 2.9 \implies 7 - 28b + 42b = 8.7 \implies 14b = 1.7 \implies b = \frac{17}{140}$$ **(M1)**

$$a = \frac{1 - 4 \times \frac{17}{140}}{3} = \frac{72/140}{3} = \frac{6}{35}$$

$a = \dfrac{6}{35} \approx 0.171, \quad b = \dfrac{17}{140} \approx 0.121$ **A1**

$$P(X \geq 3) = 2a + 3b = \frac{12}{35} + \frac{51}{140} = \frac{48}{140} + \frac{51}{140} = \frac{99}{140} \approx 0.707$$ **A1**

Note: Accept exact fractions or 3 s.f. decimals. Both a and b must be positive for the final A1A1 to be awarded.

Q23 — Expected gain and minimum unfavourable ticket price HARD

Sample space and gains (each face has probability $\frac{1}{6}$):
Odd results $\{1,3,5\}$: gain $= 3 \times \{1,3,5\} = \{3,9,15\}$.
Even results $\{2,4,6\}$: gain $= -\{2,4,6\} = \{-2,-4,-6\}$. **M1**
Expected gain per game (before any ticket cost):

$$E(\text{gain}) = \frac{1}{6}(3+9+15-2-4-6) = \frac{1}{6}(15) = \frac{5}{2} = 2.5 \text{ dollars}$$

A1

Note: This is positive, so the game favours the player before any ticket price is charged.

Net gain with ticket price t:

$$E(\text{net}) = \frac{5}{2} - t$$

M1

The game is unfavourable for the player when $E(\text{net}) < 0$, i.e. $\frac{5}{2} - t < 0$, i.e. $t > 2.5$. Checking whole-number prices: at $t = 2$, $E(\text{net}) = 0.5 \geq 0$ (favourable); at $t = 3$, $E(\text{net}) = -0.5 < 0$ (unfavourable). **A1**

The minimum whole-number price is $t = \$3$. **R1**

Note: R1 requires explicit comparison of $t = 2$ and $t = 3$ against the threshold 2.5. Accept equivalent reasoning.

Q24 — Conditional probability from a constructed distribution HARD

(a) Values: $P(X = -1) = c(4 - 1) = 3c$, $P(X = 0) = 4c$, $P(X = 1) = 3c$ **M1**

$3c + 4c + 3c = 1 \implies 10c = 1 \implies c = \frac{1}{10}$ **A1**

(b)

$$P(X \geq 0) = P(X = 0) + P(X = 1) = \frac{4}{10} + \frac{3}{10} = \frac{7}{10}$$

M1

$$P(X = 1 \mid X \geq 0) = \frac{P(X = 1)}{P(X \geq 0)} = \frac{3/10}{7/10} = \frac{3}{7}$$

M1A1

Note: M1 in (b) for the correct conditional probability structure with numerator and denominator identified from the distribution.

Q25 — Expected absolute difference from a sample space HARD

Sample space: $5 \times 5 = 25$ equally likely outcomes. **M1**

Distribution of $X = |\text{difference}|$:

$$P(X = 0) = \frac{5}{25},\ P(X = 1) = \frac{8}{25},\ P(X = 2) = \frac{6}{25},\ P(X = 3) = \frac{4}{25},\ P(X = 4) = \frac{2}{25}$$

A1

(Difference 0: pairs $(1,1), (2,2), (3,3), (4,4), (5,5)$; diff 1: 8 pairs; diff 2: 6; diff 3: 4; diff 4: 2.)

$$E(X) = 0 \cdot \frac{5}{25} + 1 \cdot \frac{8}{25} + 2 \cdot \frac{6}{25} + 3 \cdot \frac{4}{25} + 4 \cdot \frac{2}{25}$$

M1

$$= \frac{0 + 8 + 12 + 12 + 8}{25} = \frac{40}{25}$$

(A1)

$$= \frac{8}{5} = 1.6$$

A1

Note: Award (A1) for the correct numerator 40; final A1 for $\frac{8}{5}$ or 1.6. Accept 1.60.

Q26 — Root rejection in geometric-type probability HARD

Setting up the equation: $P(X = 1) = p$ and $P(X = 3) = (1-p)^2 p$ **M1**

Given condition: $p = 3(1-p)^2 p$ **M1**

Since $0 < p < 1$, $p \neq 0$, so dividing both sides by p is valid:

$$1 = 3(1-p)^2$$

A1

$$(1-p)^2 = \frac{1}{3} \implies 1 - p = \pm\frac{1}{\sqrt{3}} \implies p = 1 \mp \frac{1}{\sqrt{3}}$$

$$p = 1 - \frac{\sqrt{3}}{3} \quad \text{or} \quad p = 1 + \frac{\sqrt{3}}{3}$$

A1

Since $0 < p < 1$: the root $p = 1 + \frac{\sqrt{3}}{3} \approx 1.577$ is rejected, as it exceeds 1. **R1**

$$p = 1 - \frac{\sqrt{3}}{3} = \frac{3 - \sqrt{3}}{3} \approx 0.423 \text{ (3 s.f.)}$$

A1

Note: R1 requires explicit rejection of the invalid root with a stated reason. Award R1 only if both roots are identified and the rejected root's reason is given.

Q27 — Show a equals b via simultaneous equations HARD

Sum to 1: $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$

$a + a + b + b = 1 \implies 2a + 2b = 1$ **M1**

Expected value: $E(X) = 0 \cdot a + 1 \cdot a + 2 \cdot b + 3 \cdot b = a + 5b = \frac{3}{2}$ **M1**

Solving the system: from $2a + 2b = 1$: $a = \frac{1}{2} - b$ **A1**

Substituting: $\left(\frac{1}{2} - b\right) + 5b = \frac{3}{2} \implies \frac{1}{2} + 4b = \frac{3}{2} \implies 4b = 1 \implies b = \frac{1}{4}$ **A1**

$a = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so $a = b = \frac{1}{4}$. **AG**

Note: Both equations must be shown before solving. Final AG line earns nothing.

Q28 — Net gain distribution and a fairness decision HARD

Perfect squares in $\{1, \ldots, 20\}$: $\{1, 4, 9, 16\}$, so 4 values.
Even non-squares: $\{2, 6, 8, 10, 12, 14, 18, 20\}$ (excluding 4, 16), so 8 values.
Otherwise: remaining $20 - 4 - 8 = 8$ values. **M1**
Net gains (after \$5 ticket): win \$50 gives net $G = 45$; win \$10 gives $G = 5$; win nothing gives $G = -5$. **A1**

$$E(G) = 45 \cdot \frac{4}{20} + 5 \cdot \frac{8}{20} + (-5) \cdot \frac{8}{20}$$

M1

$$= \frac{180 + 40 - 40}{20} = \frac{180}{20} = 9 \text{ dollars}$$

A1

Since $E(G) = 9 > 0$, the game favours the player and is not fair. **R1**

Note: R1 requires explicit comparison of $E(G)$ with zero and a stated conclusion. Trap: students may forget to subtract the \$5 ticket price from winnings.

Q29 — Conditional probability from a table with unknown k HARD

(a) $0.1 + k + 0.3 + 2k + 0.1 = 1$ **M1**
$3k = 0.5 \implies k = \frac{1}{6} \approx 0.167$ **A1**

(b) $P(X > 0) = 1 - P(X = 0) = 1 - 0.1 = 0.9$ **M1**
$P(1 \le X \le 3 \text{ and } X > 0) = P(X = 1) + P(X = 2) + P(X = 3) = k + 0.3 + 2k = 3k + 0.3$
$= 3 \times \frac{1}{6} + 0.3 = 0.5 + 0.3 = 0.8$ **M1**

$$P(1 \le X \le 3 \mid X > 0) = \frac{0.8}{0.9} = \frac{8}{9} \approx 0.889$$

A1

Note: Award M1 in (b) for the correct conditional structure. Accept $\frac{8}{9}$ or 0.889 (3 s.f.).

Q30 — Expected value from a two-stage experiment HARD

Setting up outcomes: $P(\text{first roll} = j) = \frac{1}{4}$ for $j \in \{1, 2, 3, 4\}$. **M1**
If first roll $\in \{2, 3, 4\}$: $X = j$ with probability $\frac{1}{4}$ each.

If first roll $= 1$: second roll $r \in \{1, 2, 3, 4\}$ each with probability $\frac{1}{4}$, so $X = 1 + r \in \{2, 3, 4, 5\}$ each with probability $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$. **A1**

Computing $E(X)$:

$$E(X) = \frac{1}{4}(2 + 3 + 4) + \frac{1}{16}(2 + 3 + 4 + 5)$$

M1

$$= \frac{9}{4} + \frac{14}{16} = \frac{9}{4} + \frac{7}{8} = \frac{18}{8} + \frac{7}{8} = \frac{25}{8}$$

A1

$$= 3.125$$

A1

Note: First A1 for correctly identifying all branches and their probabilities. Second M1 for a complete sum; accept equivalent grouping. Final A1 for $\frac{25}{8}$ or 3.125 or 3.13 (3 s.f.).