# IB Math AA SL — Topic 4

## Probability Distributions

Discrete Probability Distributions · Expected Values

| | | | |
|---|---|---|---|
| **Difficulty** | EASY | MEDIUM | HARD |
| **Questions** | 10 | 10 | 10 |
| **Total Marks** | 30 | 40 | 50 |

Name:______________________________ Date:_______________

Score:__________/120

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**Instructions:** Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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### Key Formulae

- **Sum of all probabilities:** $\sum_x P(X = x) = 1$
- **Expected value:** $\mathrm{E}(X) = \sum_x x \cdot P(X = x)$
- **Expected value of a function:** $\mathrm{E}(g(X)) = \sum_x g(x) \cdot P(X = x)$
- **Conditional probability:** $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
- **Expected gain in a game (stake $s$):** $\mathrm{E}(\text{gain}) = \mathrm{E}(X) - s$; fair game when $\mathrm{E}(\text{gain}) = 0$
- **Complementary probability:** $P(X \le k) = 1 - P(X > k)$
- **Fair ticket price:** set $\mathrm{E}(\text{winnings}) = \text{price}$, so price $= \sum_x x \cdot P(X = x)$

## GDC Tips

- **Storing a table:** Enter $x$-values in `List 1` and $P(X = x)$ values in `List 2`; verify the sum of `List 2` equals $1$ using `sum(L2)`.
- **Computing** $\mathrm{E}(X)$**:** Use `sum(L1*L2)` after entering values in `List 1` and `List 2`; this directly evaluates $\sum x \cdot P(X = x)$.
- **Solving for a parameter** $k$**:** Use `solve()` or the equation solver with the sum-to-1 constraint typed explicitly; check the result satisfies all probability conditions ($0 \leq P \leq 1$).
- **Computing** $\mathrm{E}(g(X))$**:** Define `List 3 = g(L1)`, then evaluate `sum(L3*L2)` for quantities such as $\mathrm{E}(X^2)$ or $\mathrm{E}(2X + 1)$.
- **Conditional probabilities:** Store numerator and denominator events as separate sums from the table; divide them directly rather than retyping fractions.
- **Two-unknown systems:** Enter both the sum-to-1 equation and the $\mathrm{E}(X)$ equation into `solve({…,…},{a,k})` simultaneously to avoid sign errors from manual substitution.

## Common Mistakes to Avoid

1. **Forgetting to verify sum-to-1.** After finding a parameter $k$, students substitute back to check $\mathrm{E}(X)$ but neglect to confirm every individual $P(X = x) \geq 0$ and that the total is exactly $1$. Any value of $k$ that makes a probability negative or greater than $1$ must be rejected even if the algebra is correct.
2. **Confusing** $\mathrm{E}(X)$ **with the fair ticket price.** The fair price equals $\mathrm{E}(\text{winnings})$, not $\mathrm{E}(\text{net gain})$. Students who subtract the stake before equating to zero arrive at a price that is off by the stake amount. Always separate gross winnings from the cost of play before equating.
3. **Incorrect conditional probability numerator.** For $P(X > s \mid X < t)$, students frequently use $P(X > s)$ in the numerator instead of $P(s < X < t)$. The numerator must be the intersection $P(X > s \text{ and } X < t)$, read directly from the table rows satisfying both inequalities simultaneously.
4. **Using** $\mathrm{E}(X^2)$ **instead of** $[\mathrm{E}(X)]^2$**.** Variance is not in AA SL, but questions on $\mathrm{E}(X^2)$ or $\mathrm{E}(2X - 3)$ do appear. Students mis-square the expected value rather than computing the expectation of the squared variable; these two quantities are generally not equal.
5. **Wrong direction in a two-unknown simultaneous system.** When two equations (sum-to-1 and a given $\mathrm{E}(X)$) are set up, students sometimes write the $\mathrm{E}(X)$ equation using the $P$-values instead of the $x$-values as multipliers, or vice versa. Write both equations in full before solving; do not compress steps.
6. **Omitting the parameter order condition.** When a question states "where $p < q$", both solutions of a symmetric system satisfy the equations algebraically, but only one respects the ordering constraint. Failing to check and declare which solution is accepted loses the final accuracy mark, even when the method is otherwise perfect.

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## Section A — Easy

**EASY**

**1.** The discrete random variable $X$ has the following probability distribution, where $k \in \mathbb{R}$.

| $x$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|
| $P(X = x)$ | $0.1$ | $k$ | $0.3$ | $2k$ |

Find the value of $k$. *Calculator not allowed* **[3 marks]**

**2.** The discrete random variable $X$ has the following probability distribution.

| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $P(X = x)$ | $0.05$ | $0.20$ | $0.35$ | $0.25$ | $0.15$ |

Find $\mathrm{E}(X)$. *Calculator allowed* **[3 marks]**

**3.** A discrete random variable $X$ takes values $1$, $2$, and $3$ with probabilities $0.4$, $0.35$, and $0.25$ respectively. Find E$(X)$. *Calculator allowed* **[3 marks]**

**4.** The discrete random variable $X$ has the following probability distribution, where $a \in \mathbb{R}$.

| $x$ | $2$ | $4$ | $6$ |
|---|---|---|---|
| $P(X = x)$ | $a$ | $3a$ | $a$ |

Find the value of $a$. *Calculator not allowed* **[3 marks]**

**5.** A discrete random variable $X$ takes values $0$, $1$, $2$, and $3$ with $P(X = 0) = 0.3$, $P(X = 1) = 0.4$, $P(X = 2) = 0.2$, and $P(X = 3) = 0.1$.
Find $P(X \leq 1)$. *Calculator allowed* **[3 marks]**

**6.** The discrete random variable $X$ has the following probability distribution, where $k \in \mathbb{R}$.

| $x$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|
| $P(X = x)$ | $2k$ | $k$ | $3k$ | $k$ |

Find the value of $k$, and hence find E$(X)$. *Calculator not allowed* **[3 marks]**

**7.** A spinner has four sectors numbered $1$, $2$, $3$, and $4$. The probability of landing on each sector is equal. Let $X$ be the number shown after one spin, where $X \in \{1, 2, 3, 4\}$.
Find E$(X)$. *Calculator allowed* **[3 marks]**

**8.** A discrete random variable $X$ takes values $0$, $5$, and $10$ with probabilities $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$ respectively.
Find E$(X)$. *Calculator not allowed* **[3 marks]**

**9.** A game gives a player a prize of $\$3$ with probability $0.2$, a prize of $\$1$ with probability $0.5$, and no prize with probability $0.3$. Let $X$ be the prize amount in dollars received by a player.
Find E$(X)$. *Calculator allowed* **[3 marks]**

**10.** A discrete random variable $X$ takes values $1$, $2$, $3$, $4$, and $5$, each with equal probability.
Find E$(X)$. *Calculator allowed* **[3 marks]**

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## Section B — Medium

MEDIUM

**11.** The discrete random variable $X$ has the following probability distribution, where $k \in \mathbb{R}$.

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X = x)$ | $2k$ | $k$ | $3k$ | $0.2$ |

Find the value of $k$. *Calculator not allowed* **[4 marks]**

**12.** The discrete random variable $X$ has the following probability distribution, where $a, b \in \mathbb{R}$.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | $0.1$ | $a$ | $b$ | $0.15$ |

Given that $\mathrm{E}(X) = 1.7$, find the values of $a$ and $b$. *Calculator allowed* **[4 marks]**

**13.** A fair six-sided die has faces numbered $1, 1, 2, 3, 4, 5$. The die is rolled once and the number showing, $X$, is recorded.

**(a)** Write down $P(X = 1)$. **[1 marks]**

**(b)** Find $\mathrm{E}(X)$. **[3 marks]** *Calculator not allowed*

**14.** The discrete random variable $X$ has the following probability distribution, where $k \in \mathbb{R}$.

| $x$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $P(X = x)$ | $0.05$ | $k$ | $0.25$ | $3k$ | $0.10$ |

**(a)** Find the value of $k$. **[2 marks]**

**(b)** Find $\mathrm{E}(X)$. **[2 marks]** *Calculator allowed*

**15.** Amara plays a game in which she rolls a fair four-sided die with faces numbered $1, 2, 3, 4$. She wins $\$5$ if the die shows a $4$, wins $\$2$ if the die shows a $3$, and loses $\$1$ otherwise.
Let $W$ be Amara's winnings, in dollars, on a single play of the game.

**(a)** Write down all possible values of $W$. **[1 marks]**

**(b)** Find $\mathrm{E}(W)$. **[3 marks]** *Calculator allowed*

**16.** The discrete random variable $X$ has the following probability distribution, where $k \in \mathbb{R}^+$.

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $P(X = x)$ | $k^2$ | $2k$ | $0.16$ |

Find the value of $k$. *Calculator not allowed* **[4 marks]**

**17.** Beto records the number of text messages, $X$, he receives each morning before 9 am. The probability distribution of $X$ is given below, where $a \in \mathbb{R}$.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | $0.3$ | $0.4$ | $a$ | $0.05$ |

**(a)** Find the value of $a$. **[1 marks]**
**(b)** Find $P(X \leq 2 \mid X \geq 1)$. **[3 marks]** *Calculator al-*

*lowed*

**18.** A charity raffle sells $200$ tickets at $\$3$ each. There is one prize of $\$150$ and two prizes of $\$50$. Chiara buys one ticket. Let $G$ be Chiara's net gain, in dollars, from buying the ticket.
**(a)** Write down the three possible values of $G$. **[1 marks]**
**(b)** Find E$(G)$. **[3 marks]** *Calculator allowed*

**19.** The discrete random variable $X$ has probability distribution given by

$$P(X = x) = \frac{c}{x}, \quad x \in \{1,\, 2,\, 4,\, 5\},$$

where $c \in \mathbb{R}^+$.
**(a)** Find the exact value of $c$. **[2 marks]**
**(b)** Find E$(X)$, giving your answer as an exact fraction. **[2 marks]** *Calculator not allowed*

**20.** Dina plays a game by spinning a fair spinner with sectors labelled $1, 2, 3, 4, 5, 6$. She pays $\$t$ to play. She receives $\$10$ if the spinner lands on $6$, $\$4$ if it lands on $5$, and nothing otherwise.
**(a)** Find E$(W)$, the expected winnings (before subtracting the cost), in dollars. **[2 marks]**
**(b)** Find the value of $t$ for which the game is fair. **[2 marks]** *Calculator allowed*

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## Section C — Hard

**HARD**

**21.** The discrete random variable $X$ has the following probability distribution, where $k \in \mathbb{R}$, $k > 0$. *Calculator not allowed*

**[5 marks]**

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $P(X = x)$ | $k^2$ | $2k$ | $3k$ | $2k$ | $k^2$ |

Show that $k = \dfrac{-7 + \sqrt{57}}{4}$.

**22.** A discrete random variable $X$ takes values $1, 2, 3, 4$ with probabilities as shown in the table below, where $a, b \in \mathbb{R}$, $a > 0$, $b > 0$. *Calculator allowed* **[5 marks]**

| $x$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|
| $P(X = x)$ | $a$ | $b$ | $2a$ | $3b$ |

Given that $\text{E}(X) = 2.9$, find the values of $a$ and $b$, and hence find $P(X \geq 3)$.

**23.** Riku plays a game in which he rolls a fair six-sided die. If the result is odd he wins that number of dollars. If the result is even he loses that number of dollars. *Calculator allowed* **[5 marks]**
Find the expected gain per game, and hence determine the minimum whole-number ticket price (in dollars) that makes the game unfavourable for the player.

**24.** The discrete random variable $X$ has $P(X = x) = c(4 - x^2)$ for $x \in \{-1, 0, 1\}$, where $c \in \mathbb{R}$, $c > 0$. *Calculator not allowed* **[5 marks]**

(a) Find the value of $c$. [2]

(b) Find $P(X = 1 \mid X \geq 0)$. [3]

**25.** Yuna and Dmitri each independently select a single integer at random from the set $\{1, 2, 3, 4, 5\}$. Let $X$ be the absolute difference between their two chosen numbers. *Calculator allowed* **[5 marks]**

Find E$(X)$.

**26.** A biased coin has probability $p$ of showing heads, where $0 < p < 1$. The coin is tossed repeatedly until a head appears. Let $X$ be the number of tosses required. It is given that $P(X = 1) = P(X = 3)$. *Calculator not allowed* **[5 marks]**

Find the value of $p$, clearly rejecting any values that are not valid.

**27.** The discrete random variable $X$ takes values in $\{0, 1, 2, 3\}$. The probabilities satisfy $P(X = 0) = P(X = 1) = a$ and $P(X = 2) = P(X = 3) = b$, where $a, b > 0$. *Calculator not allowed* **[5 marks]**

Given that $\mathrm{E}(X) = \frac{3}{2}$, show that $a = b = \frac{1}{4}$.

**28.** Fatima plays a lottery game. She pays \$5 to enter. A single integer is selected uniformly at random from $\{1, 2, 3, \ldots, 20\}$. Fatima wins \$50 if the number is a perfect square, \$10 if the number is even but not a perfect square, and nothing otherwise. Let $G$ be Fatima's net gain in dollars from one game. *Calculator allowed* **[5 marks]**

Find $\mathrm{E}(G)$, and determine whether the game is fair.

**29.** The discrete random variable $X$ has the probability distribution shown, where $k \in \mathbb{R}, k > 0$. *Calculator allowed* **[5 marks]**

| $x$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|
| $P(X = x)$ | $0.1$ | $k$ | $0.3$ | $2k$ | $0.1$ |

(a) Find the value of $k$. [2]
(b) Find $P(1 \leq X \leq 3 \mid X > 0)$. [3]

**30.** The random variable $X$ is defined as follows: a fair four-sided die (faces $1, 2, 3, 4$) is rolled. If the result is greater than $1$, that value is recorded as $X$. If the result is $1$, the die is rolled a second time and the sum of both rolls is recorded as $X$. *Calculator allowed* **[5 marks]**

Find $\mathrm{E}(X)$.

## Worked Solutions

### Q1 — Sum-to-one, find $k$ [EASY]

Probabilities sum to $1$:

$$0.1 + k + 0.3 + 2k = 1$$

**M1**

$3k = 0.6 \Rightarrow k = 0.2$ **A1**

*Note: Award M1 for a correct sum-to-one equation with all four terms present. A1 for $k = 0.2$ only. Condone $k = \frac{1}{5}$.*

### Q2 — Expected value from table [EASY]

$\mathsf{E}(X) = 0(0.05) + 1(0.20) + 2(0.35) + 3(0.25) + 4(0.15)$ **M1**

$= 0 + 0.20 + 0.70 + 0.75 + 0.60$ **(A1)**

$\mathsf{E}(X) = 2.25$ **A1**

*Note: Award M1 for an attempt to evaluate $\sum x \cdot P(X = x)$ with at least three correct products. Final A1 for $2.25$.*

### Q3 — Expected value, three values [EASY]

$\mathsf{E}(X) = 1(0.4) + 2(0.35) + 3(0.25)$ **M1**

$= 0.4 + 0.70 + 0.75$ **(A1)**

$\mathsf{E}(X) = 1.85$ **A1**

*Note: Award M1 for a correct substitution into $\sum x \cdot P(X = x)$. Final A1 for $1.85$.*

### Q4 — Sum-to-one, find $a$ [EASY]

Probabilities sum to $1$:

$$a + 3a + a = 1$$

**M1**

$5a = 1 \Rightarrow a = \dfrac{1}{5}$ **A1**

*Note: Award M1 for a correct sum-to-one equation. A1 for $a = \frac{1}{5}$ or $0.2$. Accept either exact form.*

### Q5 — Cumulative probability [EASY]

$P(X \le 1) = P(X = 0) + P(X = 1)$ **M1**

$= 0.3 + 0.4$ **(A1)**

$P(X \le 1) = 0.7$ **A1**

*Note: Award M1 for identifying and summing the correct two probabilities. Final A1 for* $0.7$.

### Q6 — Sum-to-one then E$(X)$ [EASY]

$2k + k + 3k + k = 1 \Rightarrow 7k = 1 \Rightarrow k = \dfrac{1}{7}$ **M1**

$\mathsf{E}(X) = 1(\frac{2}{7}) + 2(\frac{1}{7}) + 3(\frac{3}{7}) + 4(\frac{1}{7}) = \dfrac{2+2+9+4}{7} = \dfrac{17}{7}$ **A1**

$\mathsf{E}(X) = \dfrac{17}{7}$ **A1**

*Note: First A1 for* $k = \frac{1}{7}$. *Second A1 for* E$(X) = \frac{17}{7}$. *Accept* $2\frac{3}{7}$ *or exact equivalent. Condone* $2.43$ *(3 s.f.) only if exact form seen.*

### Q7 — Uniform discrete E$(X)$ [EASY]

Each value has probability $\dfrac{1}{4}$.

$\mathsf{E}(X) = 1(\frac{1}{4}) + 2(\frac{1}{4}) + 3(\frac{1}{4}) + 4(\frac{1}{4})$ **M1**

$= \dfrac{1+2+3+4}{4} = \dfrac{10}{4}$ **(A1)**

$\mathsf{E}(X) = 2.5$ **A1**

*Note: Award M1 for correct substitution. Accept* $\frac{5}{2}$ *or* $2.5$.

### Q8 — Expected value, exact fractions [EASY]

$\mathsf{E}(X) = 0(\frac{1}{2}) + 5(\frac{1}{3}) + 10(\frac{1}{6})$ **M1**

$= 0 + \dfrac{5}{3} + \dfrac{10}{6} = \dfrac{10}{6} + \dfrac{10}{6}$ **(A1)**

$\mathsf{E}(X) = \dfrac{10}{3}$ **A1**

*Note: Award M1 for correct substitution into* $\sum x \cdot P(X = x)$. *Final A1 for* $\frac{10}{3}$. *Condone* $3\frac{1}{3}$. *Do not accept decimal approximation without exact form.*

### Q9 — Expected prize value [EASY]

$\text{E}(X) = 3(0.2) + 1(0.5) + 0(0.3)$ **M1**

$= 0.6 + 0.5 + 0$ **(A1)**

$\text{E}(X) = \$1.10$ **A1**

*Note: Award M1 for an attempt at* $\sum x \cdot P(X = x)$ *with at least two correct products. Final A1 for* $1.10$*. Accept* $1.1$.

### Q10 — Uniform discrete $\text{E}(X)$, five values [EASY]

Each value has probability $\frac{1}{5}$.

$\text{E}(X) = (1 + 2 + 3 + 4 + 5) \times \frac{1}{5}$ **M1**

$= \frac{15}{5}$ **(A1)**

$\text{E}(X) = 3$ **A1**

*Note: Award M1 for correct substitution into* $\sum x \cdot P(X = x)$*. Accept any correct grouping. Final A1 for* $3$.

### Q11 — Find $k$ from probability table [MEDIUM]

Probabilities sum to $1$: **M1**

$$2k + k + 3k + 0.2 = 1$$

$$6k = 0.8$$

$$k = \frac{0.8}{6} = \frac{2}{15} \approx 0.133$$

**A1**

*Note: Accept* $k = \frac{2}{15}$ *or any exact equivalent; condone* $0.133$ *(3 s.f.) if exact form seen.*

### Q12 — Two unknowns from sum-to-1 and $\text{E}(X)$ [MEDIUM]

Sum to $1$:

$$0.1 + a + b + 0.15 = 1 \implies a + b = 0.75$$ **(M1)**

Expected value equation:

$$0(0.1) + 1(a) + 2(b) + 3(0.15) = 1.7 \implies a + 2b = 1.25$$ **M1**

Subtracting: $b = 0.5$, hence $a = 0.25$ **A1A1**

*Note: Award A1 for each correct value. Accept answers obtained by GDC solver with evidence.*

### Q13 — Expected value of biased die [MEDIUM]

**(a)**

$$P(X=1)=\frac{2}{6}=\frac{1}{3}$$

**A1**

**(b)**

The full distribution is: $P(X=1)=\frac{2}{6}$, $P(X=2)=\frac{1}{6}$, $P(X=3)=\frac{1}{6}$, $P(X=4)=\frac{1}{6}$, $P(X=5)=\frac{1}{6}$ **(M1)**

$$\mathsf{E}(X)=1\cdot\frac{2}{6}+2\cdot\frac{1}{6}+3\cdot\frac{1}{6}+4\cdot\frac{1}{6}+5\cdot\frac{1}{6}$$

**M1**

$$=\frac{2+2+3+4+5}{6}=\frac{16}{6}=\frac{8}{3}$$

**A1**

*Note: Accept $2.67$ (3 s.f.) if $\frac{8}{3}$ seen. Do not accept $2.67$ alone.*

### Q14 — Find $k$ then $\mathsf{E}(X)$ [MEDIUM]

**(a)**

Sum to 1:

$$0.05+k+0.25+3k+0.10=1 \implies 4k=0.60 \implies k=0.15$$

**M1A1**

**(b)**

Using $k=0.15$, so $P(X=0)=0.15$, $P(X=2)=0.45$: **(M1)**

$$\mathsf{E}(X)=(-1)(0.05)+0(0.15)+1(0.25)+2(0.45)+3(0.10)$$

$$=-0.05+0+0.25+0.90+0.30=1.40$$

**A1**

*Note: Follow through their $k$ in part (b) provided it yields a valid distribution (all probabilities between 0 and 1).*

### Q15 — Expected winnings from a game [MEDIUM]

**(a)**

$W\in\{-1,\ 2,\ 5\}$ **A1**

**(b)**

Probabilities: $P(W=-1)=\frac{2}{4}=0.5$, $P(W=2)=\frac{1}{4}=0.25$, $P(W=5)=\frac{1}{4}=0.25$ **M1**

$$\mathsf{E}(W) = (-1)(0.5) + 2(0.25) + 5(0.25)$$

**(M1)**

$$= -0.5 + 0.5 + 1.25 = 1.25$$

**A1**

*Note: Accept \$1.25. Award M1 for correct probability assignment and (M1) for correct* $\mathsf{E}(W)$ *structure.*

**Q16 — Quadratic equation for $k$ from sum-to-1 [MEDIUM]**

Sum to 1:

$$k^2 + 2k + 0.16 = 1 \implies k^2 + 2k - 0.84 = 0$$

**M1**

Multiply by 25: $25k^2 + 50k - 21 = 0$, or use the quadratic formula:

$$k = \frac{-2 \pm \sqrt{4 + 4(0.84)}}{2} = \frac{-2 \pm \sqrt{7.36}}{2}$$

**M1**

$$k = \frac{-2 + \sqrt{7.36}}{2}$$

Since $k \in \mathbb{R}^+$, the negative root is rejected (trap: root rejection).

$$k = \frac{-2 + \sqrt{7.36}}{2} = 0.36$$

**A1**

Verification: $0.36^2 + 2(0.36) + 0.16 = 0.1296 + 0.72 + 0.16 = 1.0096\ldots$
Using exact factoring: $(k + 0.84 \cdot \frac{25}{25})\ldots$ or noting $k = 0.36$ gives $0.1296 + 0.72 + 0.16 = 1.0096$; correct value is $k = 0.3$ (see below). **A1**
*Correction:* $25k^2 + 50k - 21 = 0 \Rightarrow (5k + 7)(5k - 3) = 0 \Rightarrow k = \frac{3}{5} \cdot \frac{1}{5}$...
Using $(5k - 3)(5k + 7) = 0$: $k = \frac{3}{5} \div 5 = \frac{3}{5}$...
Factoring $k^2 + 2k - 0.84 = 0$: $(k + 2.8)(k - 0.3) = 0$, so $k = 0.3$ (since $k > 0$). **A1**
*Note: Award first M1 for forming sum-to-1 equation. Award second M1 for valid method to solve the quadratic. Final A1 for $k = 0.3$ only; the negative root $k = -2.8$ must be rejected. Award A1 (implied) for factorisation $(k + 2.8)(k - 0.3) = 0$ or equivalent quadratic formula application.*

**Q17 — Find $a$ and conditional probability [MEDIUM]**

**(a)**
$0.3 + 0.4 + a + 0.05 = 1 \implies a = 0.25$ **A1**
**(b)**

$$P(X \leq 2 \mid X \geq 1) = \frac{P(1 \leq X \leq 2)}{P(X \geq 1)}$$

**M1**

Numerator: $P(X = 1) + P(X = 2) = 0.4 + 0.25 = 0.65$ **(M1)**
Denominator: $P(X \geq 1) = 0.4 + 0.25 + 0.05 = 0.70$

$$P(X \leq 2 \mid X \geq 1) = \frac{0.65}{0.70} = \frac{13}{14} \approx 0.929$$

**A1**

*Note: Accept* $\frac{13}{14}$ *or* $0.929$ *(3 s.f.). Follow through their* $a$ *if* $0 < a < 0.3$.

## Q18 — Expected net gain from raffle ticket [MEDIUM]

**(a)**
Net gain values: $G \in \{147, 47, -3\}$ **A1**
*Note: Award A1 for all three correct values. Winnings minus the $3 ticket cost.*
**(b)**

$$P(G = 147) = \frac{1}{200}, \quad P(G = 47) = \frac{2}{200}, \quad P(G = -3) = \frac{197}{200}$$

**M1**

$$\mathsf{E}(G) = 147 \cdot \frac{1}{200} + 47 \cdot \frac{2}{200} + (-3) \cdot \frac{197}{200}$$

**(M1)**

$$= \frac{147 + 94 - 591}{200} = \frac{-350}{200} = -1.75$$

**A1**

*Note: Accept* $-\$1.75$. *Condone writing* $\mathsf{E}(G) = -1.75$ *without the dollar sign.*

## Q19 — Exact $c$ and $\mathsf{E}(X)$ from formula [MEDIUM]

**(a)**
Sum to $1$:

$$\frac{c}{1} + \frac{c}{2} + \frac{c}{4} + \frac{c}{5} = 1$$

**M1**

$$c\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5}\right) = 1 \implies c \cdot \frac{20 + 10 + 5 + 4}{20} = 1 \implies c \cdot \frac{39}{20} = 1$$

$$c = \frac{20}{39}$$

**A1**

**(b)**

$$\mathsf{E}(X) = 1 \cdot \frac{20}{39} + 2 \cdot \frac{10}{39} + 4 \cdot \frac{5}{39} + 5 \cdot \frac{4}{39}$$

**M1**

$$= \frac{20 + 20 + 20 + 20}{39} = \frac{80}{39}$$

**A1**

*Note: Award M1 for substituting their $c$ into $\sum x \cdot P(X = x)$ with at least two correct terms.*

## Q20 — Expected winnings and fair ticket price [MEDIUM]

**(a)**

$$\mathsf{E}(W) = 10 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 0 \cdot \frac{4}{6}$$

**M1**

$$= \frac{10 + 4}{6} = \frac{14}{6} = \frac{7}{3} \approx 2.33\ (\$)$$

**A1**

**(b)**
For a fair game, the cost equals the expected winnings: **M1**

$$t = \frac{7}{3} \approx 2.33$$

**A1**

*Note: Accept $t = \frac{7}{3}$ or $2.33$ (2 d.p.) since money context. Award M1 for setting $t = \mathsf{E}(W)$ or equivalent fair-game condition. Follow through their $\mathsf{E}(W)$ from part (a).*

## Q21 — Show-that: solving a quadratic for $k$ [HARD]

**Sum of probabilities equals 1:**

$k^2 + 2k + 3k + 2k + k^2 = 1$ **M1**

$2k^2 + 7k - 1 = 0$ **A1**

**Applying the quadratic formula:**

$k = \dfrac{-7 \pm \sqrt{49 + 8}}{4} = \dfrac{-7 \pm \sqrt{57}}{4}$ **M1**

Since $k > 0$, the solution $k = \dfrac{-7 - \sqrt{57}}{4}$ is rejected. **R1**

Therefore $k = \dfrac{-7 + \sqrt{57}}{4}$. **AG**

*Note: Final AG line earns nothing; R1 requires explicit reason that negative root is rejected because $k > 0$.*

## Q22 — Two-unknown distribution with E$(X)$ condition [HARD]

**Sum to 1:** $a + b + 2a + 3b = 1 \Rightarrow 3a + 4b = 1$ **M1**

**Expected value equation:**

$1 \cdot a + 2 \cdot b + 3 \cdot 2a + 4 \cdot 3b = 2.9$

$7a + 14b = 2.9$ **A1**

**Solving simultaneously:** from $3a + 4b = 1$: $a = \dfrac{1 - 4b}{3}$

$7 \cdot \dfrac{1 - 4b}{3} + 14b = 2.9 \Rightarrow 7 - 28b + 42b = 8.7 \Rightarrow 14b = 1.7 \Rightarrow b = \dfrac{17}{140}$ **(M1)**

$$a = \frac{1 - 4 \times \frac{17}{140}}{3} = \frac{1 - \frac{68}{140}}{3} = \frac{\frac{72}{140}}{3} = \frac{72}{420} = \frac{6}{35}$$

$a = \dfrac{6}{35} \approx 0.171, \quad b = \dfrac{17}{140} \approx 0.121$ **A1**

$P(X \geq 3) = 2a + 3b = \dfrac{12}{35} + \dfrac{51}{140} = \dfrac{48}{140} + \dfrac{51}{140} = \dfrac{99}{140} \approx 0.707$ **A1**

*Note: Accept exact fractions or 3 s.f. decimals. Both $a$ and $b$ must be positive for final A1A1 to be awarded.*

## Q23 — Expected gain and fair ticket price [HARD]

**Sample space and gains:**

Wins: $+1, +3, +5$ (probability $\frac{1}{6}$ each); Losses: $-2, -4, -6$ (probability $\frac{1}{6}$ each) **M1**

**Expected gain per game (before ticket):**

E(gain) $= \dfrac{1}{6}(1 + 3 + 5 - 2 - 4 - 6) = \dfrac{1}{6}(-3) = -\dfrac{1}{2}$ dollars **A1**

*Note: This is already negative, so the game without a ticket already favours the house.*

**Net gain with ticket price $t$:**

E$(G) = -\dfrac{1}{2} - t$ **M1**

The game is unfavourable for the player when E$(G) < 0$, i.e. $-\dfrac{1}{2} - t < 0$, which holds for all $t > 0$. The minimum whole-number price is $t =$ **1** dollar. **A1**

**Justification:** Since E(gain) $= -0.5 < 0$ without any ticket cost, charging any positive ticket price further decreases expected return; the minimum whole-number entry fee is $\$1$. **R1**

*Note: R1 requires an explicit comparison. Accept equivalent reasoning.*

### Q24 — Conditional probability from constructed distribution [HARD]

**(a)**

Values: $P(X=-1)=c(4-1)=3c$, $P(X=0)=4c$, $P(X=1)=3c$ **M1**

$3c+4c+3c=1 \Rightarrow 10c=1 \Rightarrow c=\frac{1}{10}$ **A1**

**(b)**

$P(X\geq 0)=P(X=0)+P(X=1)=\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$ **M1**

$P(X=1 \mid X\geq 0)=\frac{P(X=1)}{P(X\geq 0)}=\frac{\frac{3}{10}}{\frac{7}{10}}$ **M1**

$=\frac{3}{7}$ **A1**

*Note: M1 in (b) for correct conditional probability structure with numerator and denominator identified from the distribution.*

### Q25 — Expected absolute difference from a constructed sample space [HARD]

**Sample space:** $5\times 5=25$ equally likely outcomes. **M1**

**Distribution of $X=|$difference$|$:**

$P(X=0)=\frac{5}{25},\quad P(X=1)=\frac{8}{25},\quad P(X=2)=\frac{6}{25},\quad P(X=3)=\frac{4}{25},\quad P(X=4)=\frac{2}{25}$ **A1**

*(Differences 0: pairs $(1,1),(2,2),(3,3),(4,4),(5,5)$; Diff 1: 8 pairs; Diff 2: 6; Diff 3: 4; Diff 4: 2)*

$\mathrm{E}(X)=0\cdot\frac{5}{25}+1\cdot\frac{8}{25}+2\cdot\frac{6}{25}+3\cdot\frac{4}{25}+4\cdot\frac{2}{25}$ **M1**

$=\frac{0+8+12+12+8}{25}=\frac{40}{25}$ **(A1)**

$=\frac{8}{5}=1.6$ **A1**

*Note: Award (A1) for correct numerator 40; final A1 for $\frac{8}{5}$ or $1.6$. Accept $1.60$.*

### Q26 — Root rejection in geometric-type probability [HARD]

**Setting up the equation:**

$P(X=1)=p$ and $P(X=3)=(1-p)^2p$ **M1**

Setting equal: $p=(1-p)^2p$ **M1**

Since $p>0$, divide both sides by $p$:

$1 = (1-p)^2$ **A1**

$1 - p = \pm 1$

$p = 0$ or $p = 2$ **A1**

Both values are rejected: $p = 0$ violates $p > 0$; $p = 2$ violates $p < 1$ (not a valid probability). **R1**

*Note: No valid solution exists for this equation given $0 < p < 1$. R1 requires explicit rejection of both roots with stated reasons. Award R1 only if both roots are identified and both reasons given. If a candidate finds only $p = 2$ and rejects it, award at most (R1).*

### Q27 — Show that $a = b = \frac{1}{4}$ via simultaneous equations [HARD]

**Sum to 1:**

$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1$

$a + a + b + b = 1 \Rightarrow 2a + 2b = 1$ **M1**

**Expected value:**

$\mathrm{E}(X) = 0 \cdot a + 1 \cdot a + 2 \cdot b + 3 \cdot b = a + 5b = \dfrac{3}{2}$ **M1**

**Solving the system:** from $2a + 2b = 1$: $a = \dfrac{1}{2} - b$ **A1**

Substituting: $\left(\dfrac{1}{2} - b\right) + 5b = \dfrac{3}{2} \Rightarrow \dfrac{1}{2} + 4b = \dfrac{3}{2} \Rightarrow 4b = 1 \Rightarrow b = \dfrac{1}{4}$ **A1**

$a = \dfrac{1}{2} - \dfrac{1}{4} = \dfrac{1}{4}$, so $a = b = \dfrac{1}{4}$. **AG**

*Note: Both equations must be shown before solving. Final AG line earns nothing.*

### Q28 — Net gain distribution and fairness decision [HARD]

**Perfect squares in $\{1, \ldots, 20\}$: $\{1, 4, 9, 16\}$, so 4 values.**
**Even non-squares: even numbers $\{2, 6, 8, 10, 12, 14, 18, 20\}$ (excluding $4, 16$), so 8 values.**
**Otherwise: remaining $20 - 4 - 8 = 8$ values.** **M1**

**Net gains** (after \$5 ticket): win \$50 gives net $G = 45$; win \$10 gives $G = 5$; win nothing gives $G = -5$. **A1**

$\mathrm{E}(G) = 45 \cdot \dfrac{4}{20} + 5 \cdot \dfrac{8}{20} + (-5) \cdot \dfrac{8}{20}$ **M1**

$= \dfrac{180 + 40 - 40}{20} = \dfrac{180}{20} = 9$ dollars **A1**

Since $\mathrm{E}(G) = 9 > 0$, the game favours the player and is **not fair**. **R1**

*Note: R1 requires explicit comparison of $\mathrm{E}(G)$ with zero and a conclusion stated. Trap: students may forget to subtract the \$5 ticket price from winnings.*

## Q29 — Conditional probability from table with unknown $k$ [HARD]

**(a)**

$0.1 + k + 0.3 + 2k + 0.1 = 1$ **M1**

$3k = 0.5 \Rightarrow k = \dfrac{1}{6} \approx 0.167$ **A1**

**(b)**

$P(X > 0) = 1 - P(X = 0) = 1 - 0.1 = 0.9$ **M1**

$P(1 \leq X \leq 3$ and $X > 0) = P(X = 1) + P(X = 2) + P(X = 3) = k + 0.3 + 2k = 3k + 0.3$

$= 3 \times \dfrac{1}{6} + 0.3 = 0.5 + 0.3 = 0.8$ **M1**

$P(1 \leq X \leq 3 \mid X > 0) = \dfrac{0.8}{0.9} = \dfrac{8}{9} \approx 0.889$ **A1**

*Note: Award M1 in (b) for correct conditional structure. Accept $\frac{8}{9}$ or $0.889$ (3 s.f.).*

## Q30 — Expected value from a two-stage experiment [HARD]

**Setting up outcomes:**

$P(\text{first roll} = j) = \frac{1}{4}$ for $j \in \{1, 2, 3, 4\}$. **M1**

If first roll $\in \{2, 3, 4\}$: $X = j$ with probability $\frac{1}{4}$ each.

If first roll $= 1$: second roll $r \in \{1, 2, 3, 4\}$ each with probability $\frac{1}{4}$, so $X = 1 + r \in \{2, 3, 4, 5\}$ each with probability $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$. **A1**

**Computing** $\mathrm{E}(X)$**:**

$\mathrm{E}(X) = \sum_{\text{first roll}\neq 1} \frac{1}{4} \cdot j + \sum_{r=1}^{4} \frac{1}{16}(1 + r)$ **M1**

$= \frac{1}{4}(2 + 3 + 4) + \frac{1}{16}(2 + 3 + 4 + 5)$

$= \frac{9}{4} + \frac{14}{16} = \frac{9}{4} + \frac{7}{8} = \frac{18}{8} + \frac{7}{8} = \dfrac{25}{8}$ **A1**

$= 3.125$ **A1**

*Note: First A1 for correctly identifying all branches and their probabilities. Second M1 for a complete sum; accept equivalent grouping. Final A1 for $\frac{25}{8}$ or $3.125$ or $3.13$ (3 s.f.).*