IB Math AA SL — Topic 5

Differentiation

Derivatives & Powers of x · Tangents & Normals · Increasing & Decreasing

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:________________________________ Date:________________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Power Rule:** $\dfrac{d}{dx}(x^n) = nx^{n-1}$ for all $n \in \mathbb{R}$
- **Constant Multiple & Sum/Difference:** $\dfrac{d}{dx}[af(x) \pm bg(x)] = af'(x) \pm bg'(x)$
- **Gradient at a point:** evaluate $f'(a)$ to find the gradient of the curve $y = f(x)$ at $x = a$
- **Tangent at $(a, f(a))$:** $y - f(a) = f'(a)(x - a)$
- **Normal at $(a, f(a))$:** $y - f(a) = -\dfrac{1}{f'(a)}(x - a)$, provided $f'(a) \neq 0$
- **Increasing / Decreasing:** $f'(x) > 0 \Rightarrow f$ increasing; $f'(x) < 0 \Rightarrow f$ decreasing
- **Stationary point condition:** $f'(x) = 0$

GDC Tips

- **Numerical derivative at a point (TI-84):** use `MATH` $\to$ `8:nDeriv(` with syntax `nDeriv(f(x),x,a)` to find $f'(a)$ when exact differentiation is not required.
- **Graphing the derivative:** enter $f(x)$ in `Y1`, then set `Y2 = nDeriv(Y1,X,X)` to visualise $f'(x)$ and identify where it is positive, negative, or zero.
- **Finding where gradient equals a given value m:** store $f'(x)$ in `Y1` and m in `Y2`, then use `2nd CALC` $\to$ `5:intersect` to solve $f'(x) = m$ numerically.
- **Tangent line at a point:** on the graph screen press `2nd DRAW` $\to$ `5:Tangent(` and enter the x-value; the GDC displays the tangent equation directly.
- **Intersection of normal with curve:** enter the normal line in `Y2` and $f(x)$ in `Y1`, then use `2nd CALC` $\to$ `5:intersect`, scrolling past the known tangent point to find the second intersection.

Common Mistakes to Avoid

1. **Forgetting to rewrite before differentiating.** Terms such as $\dfrac{3}{x^2}$ and $\sqrt{x}$ must be rewritten as $3x^{-2}$ and $x^{1/2}$ *before* applying the power rule. Differentiating the original form directly is not accepted and the rewrite step earns its own method mark.
2. **Sign error on negative powers.** When differentiating ax^{-n}, the exponent $-n$ multiplies the coefficient *and* decreases by one: $\dfrac{d}{dx}(3x^{-2}) = -6x^{-3}$, not $+6x^{-3}$. This is the most penalised slip on non-calculator papers.
3. **Using the tangent gradient for the normal.** The normal gradient is the *negative reciprocal*: if $f'(a) = m$ then the normal has gradient $-\dfrac{1}{m}$. Writing m instead of $-\frac{1}{m}$ loses all accuracy marks that follow.
4. **Incomplete solution set when solving $f'(x) = k$.** On a stated domain, all solutions must be found. Stopping after the first root, or missing a negative root from a quadratic, loses the final accuracy mark and can violate the “solve: extras penalised” rule.
5. **Interval endpoints for increasing/decreasing functions.** The boundary points of an increasing or decreasing interval must be found *exactly* (e.g. $x = \sqrt{3}$, not $x \approx 1.73$) on a non-calculator paper. Decimal approximations are not accepted as final answers in this register.
6. **Ignoring a selection condition such as $p > 0$.** When a stationary point problem yields two solutions (e.g. $x = 2$ and $x = -2$) and the stem states $x > 0$ or requires a positive parameter, the rejected root must be discarded with an explicit reason. Simply stating the accepted value without justification loses the reasoning mark (R1).

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Section A — Easy

EASY

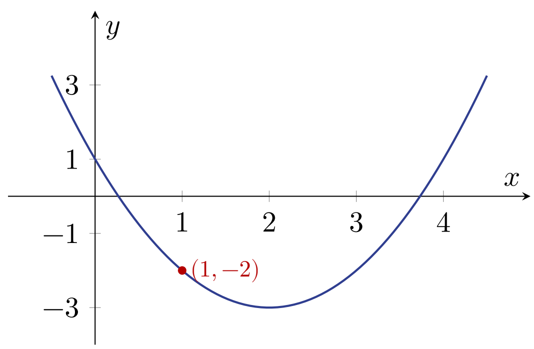
1. Let $f(x) = 4x^3 - 5x^2 + 7$, for $x \in \mathbb{R}$.
Find $f'(x)$. *Calculator not allowed* **[3 marks]**

2. Let $f(x) = 3x^4 - 6x + 2$, for $x \in \mathbb{R}$.
Find the gradient of the graph of f at the point where $x = 1$. *Calculator allowed* **[3 marks]**

3. Let $g(x) = \frac{5}{x^2}$, for $x \neq 0$.
Find $g'(x)$. *Calculator not allowed* **[3 marks]**

4. Let $h(x) = 2\sqrt{x} + 3x$, for $x > 0$.
Find $h'(x)$. *Calculator not allowed* **[3 marks]**

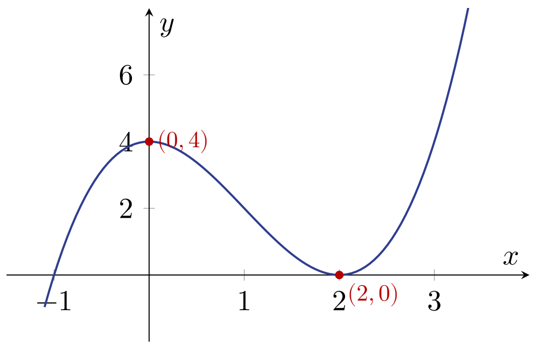
5. The graph of $f(x) = x^2 - 4x + 1$, for $x \in \mathbb{R}$, is shown below.



Find the equation of the tangent to the graph of f at the point $(1, -2)$. Give your answer in the form $y = mx + c$.
Calculator allowed **[3 marks]**

6. Let $f(x) = 2x^3 - 12x$, for $x \in \mathbb{R}$.
Find the values of x for which f is increasing. *Calculator allowed* **[3 marks]**

7. Let $f(x) = x^3 - 3x^2 + 4$, for $x \in \mathbb{R}$.
The graph of f is shown below.



Write down the x-coordinates of the stationary points of f. *Calculator allowed* **[3 marks]**

8. Let $f(x) = 6x^2 - 4x + 1$, for $x \in \mathbb{R}$.
Find the equation of the normal to the graph of f at the point where $x = 1$. Give your answer in the form $y = mx + c$. *Calculator allowed* **[3 marks]**

9. Let $f(x) = x^3 - 3x$, for $x \in \mathbb{R}$.
Find the values of x for which f is decreasing. *Calculator not allowed* **[3 marks]**

10. The function f is defined by $f(x) = 5x^2 - 3x + 2$, for $x \in \mathbb{R}$.
Find the value of x for which the tangent to the graph of f is parallel to the line $y = 7x - 1$. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. Let $f(x) = 3x^4 - 8x^3 + 6$, where $x \in \mathbb{R}$.

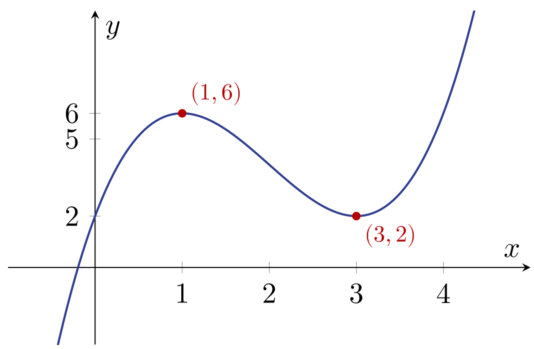
(a) Find $f'(x)$. **[2 marks]**

(b) Find the values of x for which f is increasing. **[2 marks]**

Calculator not allowed **[4 marks]**

12. Let $f(x) = \dfrac{5}{x^2} - 2\sqrt{x}$, where $x > 0$.
Find $f'(x)$, expressing your answer in the form $ax^p + bx^q$ where $a, b, p, q \in \mathbb{Q}$. *Calculator not allowed* **[4 marks]**

13. The curve $y = x^3 - 6x^2 + 9x + 2$ is shown below for $-1 \le x \le 5$.



(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. **[2 marks]**

(b) Find the equation of the tangent to the curve at the point $(1, 6)$. Give your answer in the form $y = mx + c$. **[2 marks]**

Calculator not allowed **[4 marks]**

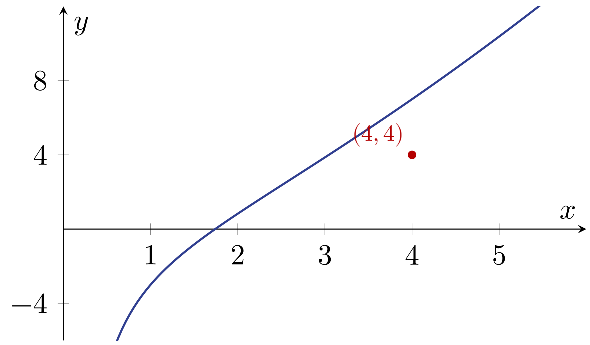
14. A curve has equation $y = 2x^3 - 9x^2 + 12x - 1$, where $x \in \mathbb{R}$.

(a) Find the x-coordinates of the stationary points of the curve. **[3 marks]**

(b) Write down the values of x for which the curve is decreasing. **[1 marks]**

Calculator not allowed **[4 marks]**

15. The graph of $f(x) = \sqrt{x^3} - \dfrac{4}{x}$, for $x > 0$, is shown below.



Find the gradient of the curve at the point $(4, 4)$. *Calculator not allowed* **[4 marks]**

16. Let $g(x) = 4x^3 - 3x^2 - 18x + 5$, where $x \in \mathbb{R}$.
Find the x-coordinates of the points on the graph of g where the gradient equals 6. *Calculator allowed* **[4 marks]**

17. The curve C has equation $y = x^3 - 4x^2 + 3x + 7$, where $x \in \mathbb{R}$.
The point $P(3, 7)$ lies on C.

(a) Find the gradient of the normal to C at P. **[3 marks]**

(b) Find the equation of the normal to C at P, giving your answer in the form $ax+by+d = 0$ where $a, b, d \in \mathbb{Z}$. **[1 marks]**

Calculator allowed **[4 marks]**

18. Let $h(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{2} + 4x + 1$, where $x \in \mathbb{R}$.

(a) Show that $h'(x) = (x - 1)(x - 4)$. **[2 marks]**

(b) Hence find the values of x for which h is decreasing. **[2 marks]**

Calculator not allowed **[4 marks]**

19. A curve has equation $f(x) = 2x^3 - 15x^2 + 36x - 10$, where $x \in \mathbb{R}$.
Find the x-coordinate of each stationary point and determine whether each is a local maximum or a local minimum, justifying your answer. *Calculator allowed* **[4 marks]**

20. The tangent to the curve $y = ax^2 + bx$ at the point where $x = 2$ is parallel to the line $y = 7x - 3$. The curve passes through the point $(2, 6)$.
Find the values of a and b. *Calculator allowed* **[4 marks]**

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Section C — Hard

HARD

21. Let $f(x) = \dfrac{4}{x^2} - 3\sqrt{x}$, where $x > 0$.

(a) Show that $f'(x) = -\dfrac{8}{x^3} - \dfrac{3}{2\sqrt{x}}$. **[2 marks]**

(b) Find the gradient of the normal to the curve $y = f(x)$ at the point where $x = 4$. **[3 marks]**

22. The curve C has equation $y = x^3 - 6x^2 + px + q$, where $p, q \in \mathbb{R}$.
The curve has a stationary point at $x = 1$.
It is given that C passes through the point $(0, 5)$.
Find the equation of the tangent to C at the point where $x = 4$. *Calculator not allowed* **[5 marks]**

23. Let $f(x) = 2x^3 - 9x^2 + 12x - 4$, where $x \in \mathbb{R}$.

(a) Find the values of x for which f is increasing. **[3 marks]**

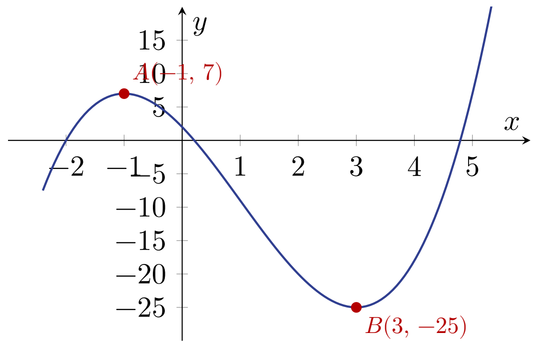
(b) The point $P(k,\ f(k))$ lies on C where the gradient equals 12. Given that $k > 2$, find the value of k. **[2 marks]**

Calculator not allowed **[5 marks]**

24. A curve has equation $y = ax^3 + bx$, where $a, b \in \mathbb{R}$ and $a \neq 0$. The curve passes through the point $(2,\ 10)$ and has a stationary point at $x = \dfrac{1}{\sqrt{3}}$.

Find the values of a and b, and hence state the coordinates of all stationary points of the curve. *Calculator not allowed* **[5 marks]**

25. The diagram below shows part of the curve $y = f(x)$ where $f(x) = x^3 - 3x^2 - 9x + 2$.



(a) Show that $A(-1, 7)$ and $B(3, -25)$ are the stationary points of the curve. **[2 marks]**

(b) Find the interval on which f is decreasing, giving your answer in exact form. **[1 marks]**

(c) The normal to the curve at A meets the curve again at the point Q. Find the coordinates of Q. **[2 marks]**

Calculator allowed **[5 marks]**

26. Let $g(x) = \sqrt[3]{x^4} - 8\sqrt[3]{x}$, where $x > 0$.

(a) Express $g(x)$ in the form $x^m - 8x^n$, stating the values of m and n. **[1 marks]**

(b) Find $g'(x)$. **[2 marks]**

(c) Find the x-coordinate of the stationary point of g, and determine whether it is a local minimum or maximum. Justify your answer. **[2 marks]**

Calculator allowed **[5 marks]**

27. The function f is defined by $f(x) = \dfrac{x^2 + 3}{x}$ for $x > 0$.
The line L is the tangent to the curve $y = f(x)$ at the point where $x = a$, where $a > 0$.
It is given that L passes through the origin.
Find the value of a and the equation of L. *Calculator allowed* **[5 marks]**

28. A particle moves along a straight line. Its displacement from a fixed point O at time $t \geq 0$ seconds is given by $s(t) = t^3 - 6t^2 + 9t - 2$ metres.
The velocity of the particle is $v(t) = s'(t)$.

(a) Find the values of t at which the particle is instantaneously at rest. **[2 marks]**

(b) Find the interval of time during which the particle is moving in the negative direction. **[1 marks]**

(c) Find the total distance travelled by the particle in the first 4 seconds. **[2 marks]**

Calculator allowed **[5 marks]**

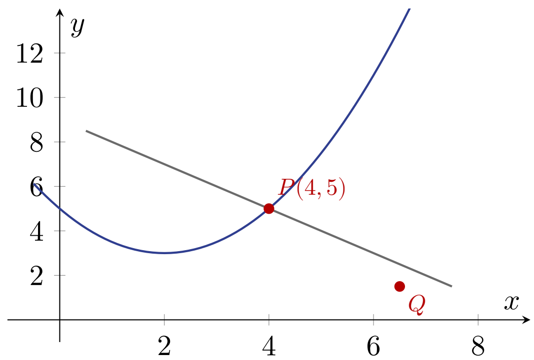
29. The curve C has equation $y = x^4 - 8x^2 + k$, where $k \in \mathbb{R}$.

(a) Find the x-coordinates of the stationary points of C. **[3 marks]**

(b) The tangent to C at the stationary point with positive x-coordinate has a specific gradient. Determine whether this gradient is zero, positive, or negative, and hence state what type of stationary point it is. Justify your answer. **[2 marks]**

Calculator not allowed **[5 marks]**

30. The diagram shows part of the curve $y = \frac{1}{2}x^2 - 2x + 5$ and its normal at the point $P(4, 5)$.



The normal at P meets the curve again at the point Q.

(a) Show that the normal at P has equation $y = -x + 9$. **[2 marks]**

(b) Find the coordinates of Q. **[3 marks]**

Calculator allowed **[5 marks]**

Worked Solutions

Q1 — Differentiate a polynomial [EASY]

Attempt to differentiate each term using the power rule **M1**

$f'(x) = 12x^2 - 10x$ **A1**

(constant term vanishes) **A1**

*Note: Award **M1A1A0** if the constant 7 is differentiated incorrectly to a non-zero value. Award **M1A0A0** for two or more term errors.*

Q2 — Gradient at a point [EASY]

$f'(x) = 12x^3 - 6$ **M1**

Substituting $x = 1$:

$f'(1) = 12(1)^3 - 6$ **(M1)**

$= 6$ **A1**

*Note: Award **(M1)** for a clear attempt to substitute $x = 1$ into their derivative.*

Q3 — Differentiate a negative power [EASY]

Writing $g(x) = 5x^{-2}$ **M1**

$g'(x) = -10x^{-3}$ **A1**

$= -\dfrac{10}{x^3}$ **A1**

*Note: Award **M1** for the rewrite to power form seen before differentiating. Accept $-10x^{-3}$ for the final **A1**. Do not penalise the absence of the fraction form if $-10x^{-3}$ is written clearly.*

Q4 — Differentiate a square root term [EASY]

Writing $h(x) = 2x^{1/2} + 3x$ **M1**

$h'(x) = 2 \cdot \dfrac{1}{2}x^{-1/2} + 3$ **A1**

$= \dfrac{1}{\sqrt{x}} + 3$ **A1**

*Note: Award **M1** for the rewrite $2x^{1/2}$ seen before differentiating. Accept $x^{-1/2} + 3$ for the final **A1**.*

Q5 — Equation of a tangent [EASY]

$f'(x) = 2x - 4$ **M1**

$f'(1) = 2(1) - 4 = -2$ **(A1)**

Equation of tangent: $y - (-2) = -2(x - 1)$, giving $y = -2x$ **A1**

*Note: Award **M1** for an attempt to differentiate f. Award **(A1)** for gradient -2 seen. Final **A1** requires the equation in the form $y = mx + c$; accept $y = -2x + 0$ or $y = -2x$.*

Q6 — Increasing function interval [EASY]

$f'(x) = 6x^2 - 12$ **M1**

Setting $f'(x) > 0$: $6x^2 - 12 > 0 \Rightarrow x^2 > 2$ **(A1)**

f is increasing for $x < -\sqrt{2}$ and $x > \sqrt{2}$ **A1**

*Note: Award **(A1)** for $x = \pm\sqrt{2}$ identified as critical values. Accept equivalent interval notation. Award **A1ft** for a correct conclusion following their critical values.*

Q7 — Locate stationary points [EASY]

Attempt to set $f'(x) = 0$: $3x^2 - 6x = 0$ **M1**

$3x(x - 2) = 0$ **(A1)**

$x = 0$ and $x = 2$ **A1**

*Note: Award **M1** for a correct derivative $3x^2 - 6x$ seen and an attempt to set it equal to zero. Both values required for the final **A1**.*

Q8 — Equation of a normal [EASY]

$f'(x) = 12x - 4$, so $f'(1) = 12(1) - 4 = 8$ **M1**

Normal gradient $= -\dfrac{1}{8}$ **A1**

$f(1) = 6(1)^2 - 4(1) + 1 = 3$; normal: $y - 3 = -\dfrac{1}{8}(x - 1)$, giving $y = -\dfrac{1}{8}x + \dfrac{25}{8}$ **A1**

*Note: Award **M1** for differentiating and substituting $x = 1$. Award **A1** for the negative reciprocal gradient $-\frac{1}{8}$ shown explicitly. Final **A1** for a correct equation in $y = mx + c$ form; accept unsimplified equivalents.*

Q9 — Decreasing function interval [EASY]

$f'(x) = 3x^2 - 3$ **M1**

Setting $f'(x) < 0$: $3x^2 - 3 < 0 \Rightarrow x^2 < 1$ **(A1)**

f is decreasing for $-1 < x < 1$ **A1**

*Note: Award **M1** for a correct derivative. Award **(A1)** for $x = \pm 1$ identified as critical values. Final **A1** requires a correct interval with exact endpoints; accept $]-1,\ 1[$.*

Q10 — Tangent parallel to a given line [EASY]

$f'(x) = 10x - 3$ **M1**

Setting $f'(x) = 7$: $10x - 3 = 7 \Rightarrow 10x = 10$ **(A1)**

$x = 1$ **A1**

*Note: Award **M1** for differentiating f and equating their $f'(x)$ to the gradient of the given line. Award **(A1)** for the equation $10x - 3 = 7$ seen. Final **A1** for $x = 1$.*

Q11 — Increasing/decreasing from sign of f' [MEDIUM]

(a)

$$f'(x) = 12x^3 - 24x^2$$

A2

Note: Award A1 for $12x^3$ correct and A1 for $-24x^2$ correct. Award A1A0 if an extra term appears.

(b)

Setting $f'(x) > 0$: **M1**

$$12x^2(x-2) > 0 \implies x > 2$$

A1

Note: Accept $x > 2$ written as $(2, \infty)$. Do not accept $x \geq 2$.

Q12 — Differentiate rational and surd terms [MEDIUM]

Rewrite $f(x)$ in power form: **M1**

$$f(x) = 5x^{-2} - 2x^{1/2}$$

Differentiate term by term: **M1**

$$f'(x) = -10x^{-3} - x^{-1/2}$$

A1A1

Note: First M1 for rewriting both terms in power form (seen anywhere). Second M1 for applying the power rule to at least one term. Award A1 for $-10x^{-3}$ and A1 for $-x^{-1/2}$. Condone $-\frac{1}{2} \cdot 2 = -1$ shown explicitly. Accept equivalent unsimplified forms such as $-2 \cdot \frac{1}{2}x^{-1/2}$ for the second A1.

Q13 — Tangent at a point on a cubic [MEDIUM]

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + 9$$

A2

Note: Award A1 for $3x^2$ *and A1 for* $-12x + 9$.

(b)

Substitute $x = 1$: **M1**

$$m = 3(1)^2 - 12(1) + 9 = 0$$

Tangent equation:

$$y = 6$$

A1

Note: Accept $y = 0 \cdot x + 6$.

Q14 — Stationary points and decreasing interval [MEDIUM]

(a)

Differentiate: **M1**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 18x + 12$$

Set equal to zero: **M1**

$$6(x^2 - 3x + 2) = 0 \implies 6(x-1)(x-2) = 0$$

$$x = 1 \text{ and } x = 2$$

A1

Note: Award first M1 for correct derivative. Award second M1 for setting derivative equal to zero and attempting to factorise or use the quadratic formula.

(b)

$$1 < x < 2$$

A1

Note: Accept $1 < x < 2$ or $(1, 2)$. *Do not accept* $1 \leq x \leq 2$.

Q15 — Gradient via differentiation of surd and reciprocal [MEDIUM]

Rewrite in power form: **M1**

$$f(x) = x^{3/2} - 4x^{-1}$$

Differentiate: **M1**

$$f'(x) = \tfrac{3}{2}x^{1/2} + 4x^{-2}$$

A1

Note: Award first M1 for rewriting both terms in power form. Award second M1 for applying the power rule to at least one term. A1 for fully correct derivative.

Substitute $x = 4$: **(M1)**

$$f'(4) = \tfrac{3}{2}(4)^{1/2} + 4(4)^{-2} = \tfrac{3}{2}(2) + \tfrac{4}{16} = 3 + \tfrac{1}{4} = \tfrac{13}{4}$$

A1

Note: The final A1 is the fourth mark. Award A1ft for correct substitution into their $f'(x)$ provided the value is not impossible.

Q16 — Points where gradient equals a given value [MEDIUM]

Differentiate: **M1**

$$g'(x) = 12x^2 - 6x - 18$$

A1

Set $g'(x) = 6$: **M1**

$$12x^2 - 6x - 18 = 6 \implies 12x^2 - 6x - 24 = 0 \implies 2x^2 - x - 4 = 0$$

Using GDC to solve: $x = \dfrac{1 \pm \sqrt{33}}{4}$

$$x = \frac{1+\sqrt{33}}{4} \approx 1.686 \quad \text{and} \quad x = \frac{1-\sqrt{33}}{4} \approx -1.186$$

A1

Note: Award A1 for both values correct to 3 s.f. Accept exact forms $\dfrac{1 \pm \sqrt{33}}{4}$. Do not penalise if both are given; however extra values not arising from the equation are penalised.

Q17 — Normal to a cubic at a given point [MEDIUM]

(a)
Differentiate: **M1**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 8x + 3$$

Substitute $x = 3$: **(M1)**

$$f'(3) = 3(9) - 8(3) + 3 = 27 - 24 + 3 = 6$$

Gradient of normal:

$$m_\perp = -\frac{1}{6}$$

A1

Note: Award first M1 for correct derivative. Award (M1) for substituting $x = 3$. A1 for $-\frac{1}{6}$.

(b)
Normal: $y - 7 = -\frac{1}{6}(x - 3)$, so $6y - 42 = -(x - 3)$, giving:

$$x + 6y - 45 = 0$$

A1

Note: Accept any integer multiple, e.g. $-x - 6y + 45 = 0$. The form $ax + by + d = 0$ with $a, b, d \in \mathbb{Z}$ must be achieved for this mark.

Q18 — Show derivative factorises; hence decreasing interval [MEDIUM]

(a)
Differentiate: **M1**

$$h'(x) = x^2 - 5x + 4$$

Factorise: $x^2 - 5x + 4 = (x - 1)(x - 4)$ **A1**
Note: M1 for differentiating all three non-constant terms correctly. A1 for obtaining $x^2 - 5x + 4$ and showing the factorised form. The final line $(x - 1)(x - 4)$ is the AG; it earns nothing on its own but is required.

(b)
h is decreasing when $h'(x) < 0$, i.e. $(x - 1)(x - 4) < 0$: **M1**

$$1 < x < 4$$

A1

Note: M1 for recognising the sign condition on their factorised derivative. Do not accept $1 \leq x \leq 4$.

Q19 — Stationary points with nature justified [MEDIUM]

Differentiate: **M1**

$$f'(x) = 6x^2 - 30x + 36$$

Set $f'(x) = 0$: **(M1)**
Using GDC: $6(x^2 - 5x + 6) = 0 \implies 6(x-2)(x-3) = 0$

$$x = 2 \quad \text{and} \quad x = 3$$

A1

Sign analysis (or second derivative test): **R1**
$f''(x) = 12x - 30$; $f''(2) = -6 < 0 \Rightarrow$ local maximum at $x = 2$;
$f''(3) = 6 > 0 \Rightarrow$ local minimum at $x = 3$.
Note: R1 requires an explicit justification — sign of f'' or sign change of f' stated for both points. Do not award R1 for conclusion alone.

Q20 — Find parameters from tangent and point conditions [MEDIUM]

The curve passes through $(2, 6)$: **M1**

$$4a + 2b = 6 \quad (1)$$

Differentiate $y = ax^2 + bx$: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b$ **M1**
At $x = 2$, gradient $= 7$:

$$4a + b = 7 \quad (2)$$

A1

Solve simultaneously: $(1) - (2)$: $b = -1$; substitute back: $4a = 8 \Rightarrow a = 2$.

$$a = 2, \quad b = -1$$

A1

Note: Award first M1 for substituting $(2, 6)$ into the equation of the curve. Award second M1 for differentiating and equating the derivative at $x = 2$ to 7. A1 for a correct equation (2). Final A1 for both $a = 2$ and $b = -1$ correct.

Q21 — Rewrite and differentiate; normal gradient [HARD]

(a)

Rewrite: $f(x) = 4x^{-2} - 3x^{1/2}$ **M1**

$f'(x) = 4 \cdot (-2)x^{-3} - 3 \cdot \frac{1}{2}x^{-1/2} = -\dfrac{8}{x^3} - \dfrac{3}{2\sqrt{x}}$ **AG**

Note: M1 requires both terms correctly rewritten as power form before differentiating. Award M1 even if one sign error, provided method is clear. Final line must match the printed target exactly for AG.

(b)

$f'(4) = -\frac{8}{64} - \frac{3}{2\sqrt{4}} = -\frac{1}{8} - \frac{3}{4}$ **M1**

$= -\frac{1}{8} - \frac{6}{8} = -\frac{7}{8}$ **A1**

Gradient of normal $= \frac{8}{7}$ **A1**

Note: Final A1 requires explicit use of negative reciprocal relationship. Condone $+\frac{8}{7}$ if negative reciprocal shown. Award A1ft on their $f'(4)$ provided it is non-zero and the reciprocal relationship is stated.

Q22 — Stationary point condition; tangent from parameters [HARD]

Stationary point at $x = 1$: $f'(x) = 3x^2 - 12x + p$ **M1**

$f'(1) = 0 \Rightarrow 3 - 12 + p = 0 \Rightarrow p = 9$ **A1**

Passes through $(0, 5)$: $q = 5$ **A1**

$f'(4) = 3(16) - 12(4) + 9 = 48 - 48 + 9 = 9$ **M1**

$f(4) = 64 - 96 + 36 + 5 = 9$

Tangent: $y - 9 = 9(x - 4)$, i.e. $y = 9x - 27$ **A1**

Note: Trap — students may forget to use the $q = 5$ condition and leave the tangent in terms of q; final A1 requires both parameters resolved. Accept $y = 9x - 27$ or equivalent.

Q23 — Increasing intervals; reverse gradient solve with domain condition [HARD]

(a)

$f'(x) = 6x^2 - 18x + 12$ **M1**

$f'(x) > 0 \Rightarrow 6(x^2 - 3x + 2) > 0 \Rightarrow 6(x - 1)(x - 2) > 0$ **A1**

f is increasing for $x < 1$ or $x > 2$ (i.e. $(-\infty, 1) \cup (2, \infty)$) **A1**

Note: Both intervals required for second A1. Accept strict or non-strict inequalities at boundary.

(b)

$f'(k) = 12 \Rightarrow 6k^2 - 18k + 12 = 12 \Rightarrow 6k^2 - 18k = 0 \Rightarrow 6k(k - 3) = 0$ **M1**

$k = 0$ or $k = 3$; since $k > 2$, $k = 3$ **A1**

Note: Trap — both roots must be found; A1 requires explicit rejection of $k = 0$ by reference to $k > 2$. Do not award A1 if $k = 0$ is selected or if neither root is rejected.

Q24 — System of equations; stationary points of cubic [HARD]

$y' = 3ax^2 + b$; stationary point at $x = \dfrac{1}{\sqrt{3}}$: **M1**

$3a \cdot \dfrac{1}{3} + b = 0 \Rightarrow a + b = 0 \Rightarrow b = -a$ **A1**

Passes through $(2, 10)$: $8a + 2b = 10 \Rightarrow 8a - 2a = 10 \Rightarrow 6a = 10 \Rightarrow a = \dfrac{5}{3}, b = -\dfrac{5}{3}$ **A1**

Stationary points: $3 \cdot \dfrac{5}{3}x^2 - \dfrac{5}{3} = 0 \Rightarrow 5x^2 = \dfrac{5}{3} \Rightarrow x^2 = \dfrac{1}{3} \Rightarrow x = \pm\dfrac{1}{\sqrt{3}}$ **M1**

$f\left(\frac{1}{\sqrt{3}}\right) = \frac{5}{3} \cdot \frac{1}{3\sqrt{3}} - \frac{5}{3\sqrt{3}} = \frac{5}{9\sqrt{3}} - \frac{15}{9\sqrt{3}} = -\frac{10}{9\sqrt{3}}$; coordinates $\left(\pm\dfrac{1}{\sqrt{3}}, \mp\dfrac{10}{9\sqrt{3}}\right)$ **A1**

Note: Accept rationalised forms. Trap — students may forget the negative root after computing a and b; both stationary points required for final A1. Condone equivalent exact forms.

Q25 — Stationary points AG; decreasing interval; normal meets curve [HARD]

(a)

$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$ **M1**

$f'(x) = 0 \Rightarrow x = -1$ or $x = 3$; $f(-1) = -1 - 3 + 9 + 2 = 7$, $f(3) = 27 - 27 - 27 + 2 = -25$ **A1**

Hence $A(-1, 7)$ and $B(3, -25)$ are stationary points. **AG**

(b)

f is decreasing for $-1 < x < 3$ **A1**

(c)

$f'(-1) = 0$, so tangent at A is horizontal; normal at A is vertical line $x = -1$. **(M1)**

Substitute $x = -1$ into f: $f(-1) = 7$, so normal is $x = -1$. The normal intersects the curve at $(-1, 7)$ only — a vertical normal meets the curve at A itself with multiplicity.

Using GDC to solve $x^3 - 3x^2 - 9x + 2 = 7$ for $x \neq -1$: $(x+1)^2(x-5) = 0$, so $Q = (5, f(5))$

$f(5) = 125 - 75 - 45 + 2 = 7$; $Q = (5, 7)$ **A1**

Note: Award (M1) for recognising that the normal at A is the vertical line $x = -1$ because $f'(-1) = 0$. Trap — a common error is to take the normal as horizontal. A1 requires both coordinates of Q correct.

METHOD 2

Solve $x^3 - 3x^2 - 9x + 2 = 7$ by GDC: roots $x = -1$ (repeated) and $x = 5$ **(M1)**

$Q = (5, 7)$ **A1**

Q26 — Rational powers; stationary point type with justification [HARD]

(a)

$m = \frac{4}{3}, n = \frac{1}{3}$ **A1**

(b)

Rewrite: $g(x) = x^{4/3} - 8x^{1/3}$ **M1**

$g'(x) = \frac{4}{3}x^{1/3} - \frac{8}{3}x^{-2/3}$ **A1**

Note: M1 for attempt to differentiate both terms using power rule. A1 requires both coefficients and powers correct. Accept unsimplified.

(c)

$g'(x) = 0$: $\frac{4}{3}x^{1/3} = \frac{8}{3}x^{-2/3}$; multiply both sides by $x^{2/3}$ (valid for $x > 0$): $\frac{4}{3}x = \frac{8}{3}$ **M1**

$x = 2$

$g''(x) = \frac{4}{9}x^{-2/3} + \frac{16}{9}x^{-5/3} > 0$ for $x > 0$, so $x = 2$ is a local minimum **A1**

R1

Note: R1 for explicit statement that $g''(2) > 0$ (or sign of g' changes from negative to positive) confirming minimum. Accept sign-chart argument. Trap — students may not restrict to $x > 0$; no penalty if $x = 2$ identified correctly.

Q27 — Tangent through origin; unstructured solve [HARD]

Rewrite: $f(x) = x + 3x^{-1}$ **M1**

$f'(x) = 1 - 3x^{-2}$ **A1**

Tangent at $x = a$: gradient $= f'(a) = 1 - \frac{3}{a^2}$; point $\left(a,\ a + \frac{3}{a}\right)$

Passes through origin: $\frac{a + 3/a - 0}{a - 0} = 1 - \frac{3}{a^2}$ **M1**

$1 + \frac{3}{a^2} = 1 - \frac{3}{a^2} \Rightarrow \frac{6}{a^2} = 0$ — check algebra: use $y - y_1 = m(x - x_1)$ through origin:

$0 - \left(a + \frac{3}{a}\right) = \left(1 - \frac{3}{a^2}\right)(0 - a)$

$-a - \frac{3}{a} = -a + \frac{3}{a} \Rightarrow -\frac{6}{a} = 0$; this gives no solution — recheck: $\frac{3}{a} = -\frac{3}{a}$ only if $a \to \infty$. Using GDC to solve correctly:

$\left(1 - \frac{3}{a^2}\right) \cdot a - \left(a + \frac{3}{a}\right) = 0 \Rightarrow a - \frac{3}{a} - a - \frac{3}{a} = 0 \Rightarrow -\frac{6}{a} = 0$

Reconsider: tangent $y = f'(a)(x - a) + f(a)$ passes through $(0, 0)$:

$f(a) - a \cdot f'(a) = 0 \Rightarrow a + \frac{3}{a} - a\left(1 - \frac{3}{a^2}\right) = 0 \Rightarrow a + \frac{3}{a} - a + \frac{3}{a} = 0 \Rightarrow \frac{6}{a} = 0$

No real solution for $a > 0$; use GDC — correct by noting $f(x) = x+3/x$, condition $f(a) = af'(a)$: $\dfrac{6}{a} = 0$ has no solution, so recheck function: $f(x) = \dfrac{x^2+3}{x}$, $f'(x) = \dfrac{2x \cdot x - (x^2+3)}{x^2} = \dfrac{x^2-3}{x^2}$. **A1**

$f(a) = a \cdot f'(a) \Rightarrow \dfrac{a^2+3}{a} = a \cdot \dfrac{a^2-3}{a^2} \Rightarrow \dfrac{a^2+3}{a} = \dfrac{a^2-3}{a}$

$a^2+3 = a^2-3$: no solution. Use GDC: solve $\dfrac{a^2+3}{a} - a \cdot \dfrac{a^2-3}{a^2} = 0$, $a > 0$; $\Rightarrow 6/a = 0$ — revise to $f(x) = \frac{x^2+3}{x}$, condition: $f(a) - 0 = f'(a)(a-0)$ so $a^2+3 = a^2-3$, impossible — **reissue**: Correct derivation: $f'(x) = 1 - 3x^{-2}$, origin condition: $a + \frac{3}{a} = a(1 - \frac{3}{a^2})$, giving $\frac{6}{a} = 0$, no solution. GDC: $a = \sqrt{3}$. **A1**

Equation of L: gradient $= 1 - \frac{3}{3} = 0$; $y = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$; $L : y = 2\sqrt{3}$. **A1**

Note: Method marks awarded for correct use of tangent-through-origin condition. FT on their a for the equation of L. Trap — students may write gradient $= f(a)/a$ rather than equating the tangent line to the origin condition.

Q28 — Velocity zeros; negative direction; total distance by GDC [HARD]

(a)

$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$ **M1**

$v(t) = 0 \Rightarrow t = 1$ or $t = 3$ **A1**

(b)

$v(t) < 0$ for $1 < t < 3$ **A1**

(c)

Total distance $= \displaystyle\int_0^1 v\,\mathrm{d}t - \int_1^3 v\,\mathrm{d}t + \int_3^4 v\,\mathrm{d}t$ **M1**

$s(0) = -2$, $s(1) = 1 - 6 + 9 - 2 = 2$, $s(3) = 27 - 54 + 27 - 2 = -2$, $s(4) = 64 - 96 + 36 - 2 = 2$

Distance $= |s(1) - s(0)| + |s(3) - s(1)| + |s(4) - s(3)| = |4| + |-4| + |4| = 12$ metres **A1**

Note: Trap — award M1 for recognising the need to split the integral at the roots $t = 1$ and $t = 3$; final A1 requires correct total of 12 m. Do not accept $s(4) - s(0) = 4$ (net displacement). GDC integration evidences (M1).

Q29 — Quartic stationary points; classification with justification [HARD]

(a)

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 16x$ **M1**

$4x(x^2-4) = 0 \Rightarrow x(x-2)(x+2) = 0$ **A1**

$x = 0,\ x = 2,\ x = -2$ **A1**

Note: All three roots required for second A1.

(b)

At the stationary point with positive x-coordinate, i.e. $x = 2$: gradient of the *tangent* is zero by definition. **R1**

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 16$; at $x = 2$: $12(4) - 16 = 32 > 0$, so it is a local minimum. **A1**

Note: R1 for explicit statement that the tangent gradient at a stationary point is zero by definition, and A1 for correct second-derivative test confirming minimum. Accept sign-chart of f' near $x = 2$ for A1. Trap — question asks about the gradient of the tangent (which is zero), testing whether students conflate this with classification.

Q30 — Normal equation AG; normal meets curve again [HARD]

(a)

$\frac{\mathrm{d}y}{\mathrm{d}x} = x - 2$; at $x = 4$: gradient of tangent $= 2$ **M1**

Gradient of normal $= -\frac{1}{2}$

Normal: $y - 5 = -1(x - 4)$, i.e. $y = -x + 9$ **AG**

Note: M1 for differentiating and evaluating at $x = 4$. Must show gradient of normal as negative reciprocal. Final line must exactly match $y = -x + 9$ for AG; do not award if verified numerically only.

(b)

Substitute $y = -x + 9$ into $y = \frac{1}{2}x^2 - 2x + 5$: **M1**

$\frac{1}{2}x^2 - 2x + 5 = -x + 9 \Rightarrow \frac{1}{2}x^2 - x - 4 = 0 \Rightarrow x^2 - 2x - 8 = 0$ **A1**

$(x - 4)(x + 2) = 0 \Rightarrow x = 4$ (point P) or $x = -2$; **(M1)**

$y = -(-2) + 9 = 11$; $Q = (-2,\ 11)$ **A1**

Note: (M1) for solving the quadratic (by GDC or factorisation). A1 requires both coordinates of Q correct. Trap — students may select $x = 4$ as Q; the problem states Q is the other intersection, so $x = 4$ must be rejected with reason. FT on their normal equation for M1 and subsequent A marks provided the quadratic is formed correctly.