IB Math AA SL — Topic 5

Further Differentiation

Chain, Product & Quotient Rules · Second Derivatives · Stationary Points, Concavity & Inflection

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Derivatives of special functions:** $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- **Chain rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$; e.g. $\frac{d}{dx}(e^{kx}) = ke^{kx}$, $\frac{d}{dx}(\ln(g(x))) = \frac{g'(x)}{g(x)}$
- **Product rule:** $\frac{d}{dx}[uv] = u'v + uv'$
- **Quotient rule:** $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}$
- **Second derivative:** $f''(x) = \frac{d^2y}{dx^2}$; stationary point at $f'(a) = 0$: maximum if $f''(a) < 0$, minimum if $f''(a) > 0$
- **Point of inflection:** $f''(a) = 0$ and f'' changes sign at $x = a$; concave up where $f''(x) > 0$, concave down where $f''(x) < 0$
- **Tangent at $x = a$:** $y - f(a) = f'(a)(x - a)$; normal gradient $= -\frac{1}{f'(a)}$

GDC Tips

- **Evaluating a derivative numerically:** use Numeric Deriv / d/dx at a given x-value to find $f'(a)$ or $f''(a)$ when exact algebra is impractical.
- **Locating stationary points:** graph $y = f'(x)$ and use Zero (or Root) to find x-values where $f'(x) = 0$; confirm with d/dx on f itself.
- **Locating inflection points:** graph $y = f''(x)$ and use Zero to find candidates; inspect the sign change of f'' either side to confirm.
- **Classifying stationary points:** evaluate $f''(a)$ using Numeric Deriv at the stationary x-value; negative $\Rightarrow$ maximum, positive $\Rightarrow$ minimum.
- **Sketching f' from f:** plot f and f' together to verify that zeros of f' align with turning points of f, and zeros of f'' align with inflection points of f.
- **Tangent and normal lines:** use the Tangent feature (some models) or manually compute $f'(a)$ numerically, then write the line equation — store $f'(a)$ in a variable to avoid rounding errors.

Common Mistakes to Avoid

1. **Wrong order in the quotient rule.** Students write $uv' - vu'$ instead of $u'v - uv'$ in the numerator. Always label u and v explicitly, differentiate each separately, then substitute into $\dfrac{u'v - uv'}{v^2}$.
2. **Forgetting the chain-rule factor.** When differentiating $e^{g(x)}$ or $\sin(g(x))$, the outer derivative is written correctly but the inner derivative $g'(x)$ is omitted. Every composite function requires multiplication by the derivative of the argument.
3. **Claiming an inflection point from $f''(a) = 0$ alone.** Setting $f''(x) = 0$ and finding a solution does not guarantee an inflection point; you must also show that f'' changes sign at that x-value. If f'' does not change sign, there is no inflection point.
4. **Misreading a graph of f' as a graph of f.** When the printed curve is labelled $y = f'(x)$, the zeros of the curve give the x-coordinates of stationary points of f, and the sign of the curve tells you where f is increasing or decreasing — not the other way around.
5. **Incorrect classification using f'': forgetting to state the evidence.** Writing "maximum" without quoting $f''(a) < 0$ loses the reasoning mark R1. Every classification must name the test used and show the numerical or algebraic evidence.
6. **Missing solutions when solving $f'(x) = 0$ on a closed interval.** Students find one stationary point algebraically but neglect to check the endpoints of the domain when determining the global maximum or minimum. Always evaluate f at every stationary point and at the endpoints.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. EASY No Calculator **[3 marks]**

Let $f(x) = e^{4x}$, for $x \in \mathbb{R}$. Find $f'(x)$.

2. EASY No Calculator **[3 marks]**

Let $g(x) = \ln(3x)$, for $x > 0$. Find $g'(x)$.

3. EASY No Calculator **[3 marks]**

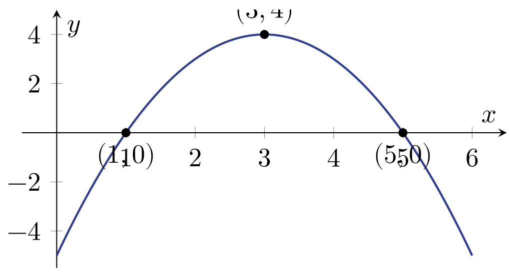
Let $h(x) = x^3 e^{2x}$, for $x \in \mathbb{R}$. Find $h'(x)$.

4. EASY No Calculator **[3 marks]**

Let $f(x) = \dfrac{\sin x}{x^2 + 1}$, for $x \in \mathbb{R}$. Find $f'(x)$.

5. EASY Calculator **[3 marks]**

The graph of $y = f(x)$ is shown below, where $f(x) = -x^2 + 6x - 5$, for $0 \leq x \leq 6$.



Find the x-coordinate of the stationary point of f and state its nature.

6. EASY Calculator **[3 marks]**

Let $f(x) = 3\sin(2x)$, for $x \in \mathbb{R}$. Find $f''(x)$.

7. EASY Calculator **[3 marks]**

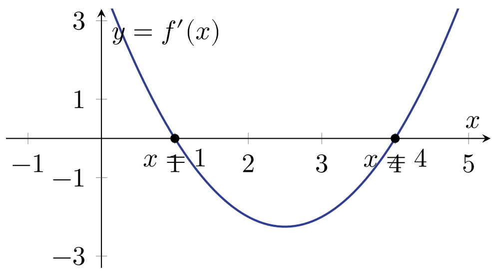
Let $f(x) = (2x + 5)^4$, for $x \in \mathbb{R}$. Find $f'(x)$.

8. EASY Calculator **[3 marks]**

The derivative of a function f is given by $f'(x) = 6x^2 - 6x$, for $x \in \mathbb{R}$. Find the x-coordinates of the stationary points of f.

9. EASY Calculator **[3 marks]**

The graph of $y = f'(x)$ is shown below for $-1 \leq x \leq 5$.



Write down the x-coordinates of any stationary points of f on $-1 \leq x \leq 5$.

10. EASY Calculator **[3 marks]**

Let $f(x) = x^4 - 12x^2$, for $x \in \mathbb{R}$. Find the x-coordinates of any points of inflection of f.

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator **[4 marks]**

Let $f(x) = e^{3x} \sin x$, where $x \in \mathbb{R}$.
Find $f'(x)$.

12. MEDIUM No Calculator **[4 marks]**

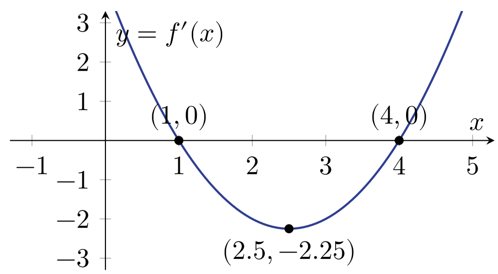
Let $g(x) = \dfrac{\ln x}{x^2}$, where $x > 0$.
Find $g'(x)$ and hence find the exact x-coordinate of the stationary point of g.

13. MEDIUM No Calculator **[4 marks]**

Let $h(x) = (2x^3 - 5)^4$, where $x \in \mathbb{R}$.
Find $h'(x)$.

14. MEDIUM Calculator **[4 marks]**

The graph of $y = f'(x)$ is shown below, where $x \in [-1, 5]$.



(a) Write down the x-values at which f has a stationary point. [1]

(b) Determine the nature of each stationary point. Justify your answers. [2]

(c) Write down the x-coordinate of the point of inflexion of f. [1]

15. MEDIUM No Calculator **[4 marks]**

Let $f(x) = x^2e^{-x}$, where $x \in \mathbb{R}$.

(a) Show that $f'(x) = e^{-x}(2x - x^2)$. [2]

(b) Find the x-coordinates of the stationary points of f. [2]

16. MEDIUM Calculator **[4 marks]**

Let $f(x) = \dfrac{3x+1}{x^2+4}$, where $x \in \mathbb{R}$.

Find the x-coordinates of the stationary points of f, giving your answers to 3 significant figures.

17. MEDIUM No Calculator **[4 marks]**

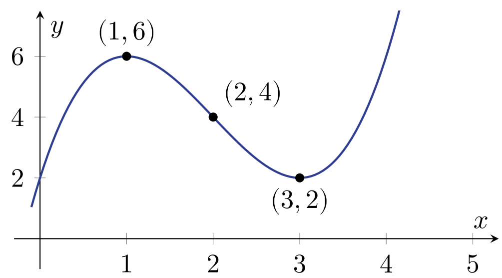
Let $f(x) = \ln(x^2+3)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [2]

(b) Find $f''(x)$. [2]

18. MEDIUM Calculator **[4 marks]**

The curve C has equation $y = x^3 - 6x^2 + 9x + 2$, where $x \in \mathbb{R}$.



(a) Find the x-coordinate of the point of inflexion of C. [2]

(b) Determine whether the stationary point at $x = 1$ is a local maximum or local minimum. Justify your answer. [2]

19. MEDIUM Calculator **[4 marks]**

Let $f(x) = \sqrt{4x^2 + 1}$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. [2]

(b) The tangent to the graph of f at the point where $x = 2$ has equation $y = mx + c$. Find the values of m and c. [2]

20. **MEDIUM** **No Calculator** **[4 marks]**

Let $f(x) = x \ln x - x$, where $x > 0$.

(a) Show that $f'(x) = \ln x$. [2]

(b) Hence find the x-coordinate and y-coordinate of the stationary point of f, giving exact values. [2]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. HARD No Calculator **[5 marks]**

Let $f(x) = x^2 e^{-3x}$, where $x \in \mathbb{R}$.

(a) Show that $f'(x) = xe^{-3x}(2 - 3x)$. [2]

(b) Find the x-coordinates of any stationary points of f, and determine the nature of each, justifying your answer using f''. [2]

(c) Find the x-coordinate of the point of inflection of f, showing that a sign change in f'' occurs. [1]

22. HARD No Calculator **[5 marks]**

Let $g(x) = \dfrac{\ln x}{x^2}$, where $x > 0$.

(a) Show that $g'(x) = \dfrac{1 - 2\ln x}{x^3}$. [2]

(b) Hence find the exact coordinates of the stationary point of g and justify that it is a maximum. [3]

23. **HARD** **No Calculator** **[5 marks]**

A curve has equation $y = \dfrac{e^{2x}}{3x-1}$, where $x \in \mathbb{R}$, $x \neq \dfrac{1}{3}$.

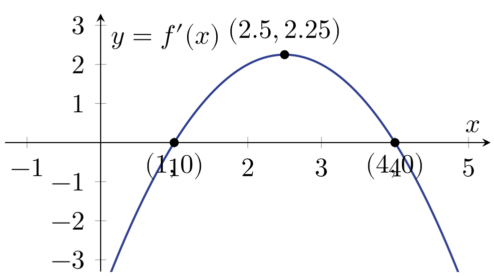
(a) Find $\dfrac{dy}{dx}$. [2]

(b) Find the exact x-coordinate of the stationary point. [2]

(c) Determine whether this stationary point is a local maximum or local minimum, justifying your answer. [1]

24. **HARD** **Calculator** **[5 marks]**

The graph of $y = f'(x)$ is shown below for $x \in [-1, 5]$.



(a) Write down the x-values at which f has a local maximum and a local minimum on $(-1, 5)$. [1]

(b) Find the x-coordinate of the point of inflection of f. [1]

(c) State the intervals on which f is concave down. [1]

(d) Given that $f(1) = 4$ and the area enclosed between $y = f'(x)$ and the x-axis from $x = 1$ to $x = 4$ is 4.5, find $f(4)$. [2]

25. HARD No Calculator **[5 marks]**

Let $h(x) = \sin(x^2 + 1)$, where $x \in \mathbb{R}$.

(a) Show that $h''(x) = 2\cos(x^2 + 1) - 4x^2 \sin(x^2 + 1)$. [3]

(b) Hence find the exact value of $h''(0)$. [1]

(c) Determine whether f is concave up or concave down at $x = 0$, justifying your answer. [1]

26. HARD Calculator **[5 marks]**

The function f is defined by $f(x) = x^3 - 6x^2 + 9x + k$, where $k \in \mathbb{R}$ and $x \in \mathbb{R}$.
It is given that f has a local minimum value of -2.
Find the value of k, the x-coordinate of the local maximum of f, and the x-coordinate of the point of inflection of f.

27. HARD No Calculator **[5 marks]**

Let $f(x) = \sqrt{4 - x^2}$, where $-2 < x < 2$.

(a) Show that $f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$. [2]

(b) Show that $f''(x) = \dfrac{-4}{(4 - x^2)^{3/2}}$. [2]

(c) Hence justify that the graph of f has no point of inflection on $-2 < x < 2$. [1]

28. HARD Calculator **[5 marks]**

The curve C has equation $y = \dfrac{\sin x}{e^x}$, where $0 \leq x \leq 2\pi$.

(a) Find $\dfrac{dy}{dx}$. [2]

(b) Find the x-coordinates of all stationary points of C on $0 \leq x \leq 2\pi$, giving your answers correct to 3 significant figures. [2]

(c) Determine the nature of each stationary point found in part (b), justifying your answer. [1]

29. HARD No Calculator **[5 marks]**

A curve is defined by $y = pxe^{-x/2}$ for $x \in \mathbb{R}$, where $p > 0$.

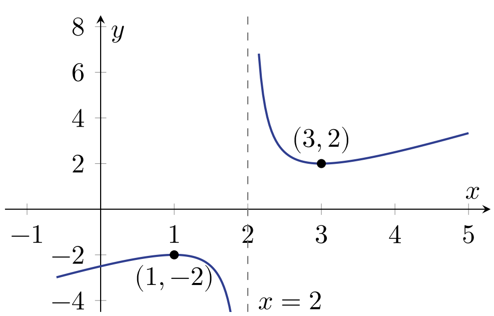
(a) Show that $\dfrac{dy}{dx} = \dfrac{pe^{-x/2}(2-x)}{2}$. [2]

(b) The tangent to the curve at the point where $x = 4$ has gradient $-e^{-2}$. Find the value of p. [2]

(c) Using your value of p, find the x-coordinate of the point of inflection of the curve. [1]

30. HARD Calculator **[5 marks]**

The function f is defined by $f(x) = \dfrac{x^2 - 4x + 5}{x - 2}$, where $x \in \mathbb{R}$, $x \neq 2$.
Find the exact coordinates of any local maximum or local minimum points of f, justifying the nature of each. Then determine whether f has any points of inflection, justifying your answer.



Mark Schemes

Q1 — Differentiating an Exponential Function EASY

Using the chain rule (or standard result) with $f(x) = e^{4x}$:

$$f'(x) = 4e^{4x}$$

M1A1

Note: Award M1 for attempt to differentiate e^{kx} giving ke^{kx}. Award A1 for the correct coefficient 4 and A1 for the retained factor e^{4x}. Condone $f' = 4e^{4x}$.

Q2 — Differentiating a Logarithm EASY

Applying the chain rule to $g(x) = \ln(3x)$:

$$g'(x) = \frac{1}{3x} \cdot 3$$

M1

$$g'(x) = \frac{1}{x}$$

A1A1

Note: Award M1 for an attempt using the chain rule. Award first A1 for $\frac{3}{3x}$. Award second A1 for the simplified answer $\frac{1}{x}$. Also accept via $\ln(3x) = \ln 3 + \ln x$ leading directly to $\frac{1}{x}$ for full marks.

Q3 — Product Rule with an Exponential Factor EASY

Applying the product rule with $u = x^3$ and $v = e^{2x}$:

$$h'(x) = 3x^2 \cdot e^{2x} + x^3 \cdot 2e^{2x}$$

M1A1

$$h'(x) = e^{2x}(3x^2 + 2x^3)$$

A1

Note: Award M1 for a correct product rule structure $u'v + uv'$. Award first A1 for both $3x^2e^{2x}$ and $2x^3e^{2x}$ correct (unsimplified). Award second A1 for the fully correct simplified form. Also accept $x^2e^{2x}(3 + 2x)$.

Q4 — Quotient Rule with a Trigonometric Numerator EASY

Applying the quotient rule with $u = \sin x$, $v = x^2 + 1$:

$$f'(x) = \frac{\cos x \cdot (x^2+1) - \sin x \cdot 2x}{(x^2+1)^2}$$

M1A1

$$f'(x) = \frac{(x^2+1)\cos x - 2x \sin x}{(x^2+1)^2}$$

A1

Note: Award M1 for a correct quotient rule structure $\frac{u'v-uv'}{v^2}$. *Award first A1 for correct numerator terms. Award second A1 for the fully correct expression. Condone omission of simplification provided the numerator and denominator are correct.*

Q5 — Stationary Point of a Quadratic EASY

Finding $f'(x)$:

$$f'(x) = -2x + 6$$

M1

Setting $f'(x) = 0$: $-2x + 6 = 0 \Rightarrow x = 3$ **A1**

Since $f''(x) = -2 < 0$, the stationary point at $x = 3$ is a maximum. **A1**

Note: Award M1 for differentiating and setting equal to zero. Award first A1 for $x = 3$. *Award second A1 for stating maximum with valid reason (e.g.* $f''(3) = -2 < 0$, *or a correct sign change of* f', *or reference to the graph). Do not award the final A1 for nature without justification.*

Q6 — Second Derivative of Sine EASY

Differentiating $f(x) = 3\sin(2x)$:

$$f'(x) = 6\cos(2x)$$

M1

$$f''(x) = -12\sin(2x)$$

A1A1

Note: Award M1 for a correct attempt to differentiate twice using the chain rule. Award first A1 for $f'(x) = 6\cos(2x)$. *Award second A1 for* $f''(x) = -12\sin(2x)$. *Condone notation* $\frac{d^2y}{dx^2}$.

Q7 — Chain Rule with a Power Function EASY

Applying the chain rule to $f(x) = (2x + 5)^4$:

$$f'(x) = 4(2x + 5)^3 \cdot 2$$

M1A1

$$f'(x) = 8(2x + 5)^3$$

A1

Note: Award M1 for an attempt at the chain rule giving $4(2x+5)^3 \times ($derivative of inner function$)$. Award first A1 for the correct unsimplified form $4(2x + 5)^3 \cdot 2$. Award second A1 for $8(2x + 5)^3$.

Q8 — Finding Stationary Points from the Derivative EASY

Setting $f'(x) = 0$:

$$6x^2 - 6x = 0$$

M1

$$6x(x - 1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

A1A1

Note: Award M1 for setting $f'(x) = 0$ and attempting to factorise or solve. Award first A1 for $x = 0$. Award second A1 for $x = 1$. Both values required for full marks.

Q9 — Reading Stationary Points from the Derivative Graph EASY

Stationary points of f occur where $f'(x) = 0$, i.e. where the graph crosses the x-axis. **R1**

From the graph: $x = 1$ **A1**

and $x = 4$ **A1**

Note: Award R1 for the correct reason that stationary points occur where $f'(x) = 0$. Award first A1 for $x = 1$. Award second A1 for $x = 4$. Both values required for the two A1 marks. Do not accept $y = 0$ without identifying the x-values.

Q10 — Points of Inflection EASY

Finding $f''(x)$:

$$f'(x) = 4x^3 - 24x, \qquad f''(x) = 12x^2 - 24$$

M1

Setting $f''(x) = 0$: $12x^2 - 24 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$ **A1**

Checking sign change: $f''(x) < 0$ for $-\sqrt{2} < x < \sqrt{2}$ and $f''(x) > 0$ outside this interval, confirming sign changes at both values. Points of inflection at $x = -\sqrt{2}$ and $x = \sqrt{2}$. **A1**

Note: Award M1 for correctly finding $f''(x) = 12x^2 - 24$ and setting equal to zero. Award first A1 for both $x = \pm\sqrt{2}$. Award second A1 for confirming a sign change in f'' at both values. Do not award the final A1 without evidence of a sign-change check.

Q11 — Product Rule with an Exponential and Sine Function MEDIUM

Recognising the product rule with $u = e^{3x}$, $v = \sin x$ **M1**

$u' = 3e^{3x}$, $v' = \cos x$ **(A1)**

$$f'(x) = 3e^{3x}\sin x + e^{3x}\cos x$$

A1

Note: Accept the equivalent factored form $e^{3x}(3\sin x + \cos x)$. Award M1 for a correct product rule structure with at least one derivative attempted correctly.

Q12 — Quotient Rule with a Logarithmic Function MEDIUM

Applying the quotient rule:

$$g'(x) = \frac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4}$$

M1

$$g'(x) = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$$

A1

Setting $g'(x) = 0$: $1 - 2\ln x = 0 \Rightarrow \ln x = \dfrac{1}{2}$ **M1**

$$x = e^{1/2} = \sqrt{e}$$

A1

Note: Award first M1 for correct quotient rule structure. Award final A1 for exact form only; $x \approx 1.65$ alone is insufficient.

Q13 — Chain Rule with a Composite Power Function MEDIUM

Recognising the chain rule with inner function $u = 2x^3 - 5$ **M1**

$$h'(x) = 4(2x^3 - 5)^3 \cdot \frac{d}{dx}(2x^3 - 5)$$

(A1)

$\frac{d}{dx}(2x^3 - 5) = 6x^2$ **(A1)**

$$h'(x) = 24x^2(2x^3 - 5)^3$$

A1

Note: Award M1 for $4(2x^3 - 5)^3$ multiplied by a derivative of the inner function. Do not penalise unsimplified forms provided the product is correct.

Q14 — Reading the Derivative Graph for Extrema and Inflexion MEDIUM

(a)
$x = 1$ and $x = 4$ **A1**

(b)
At $x = 1$: f' changes from positive to negative (the curve is above the axis before $x = 1$ and below it just after), so this is a local maximum. **R1**

At $x = 4$: f' changes from negative to positive, so this is a local minimum. **R1**

Summary: x=1: local maximum; x=4: local minimum. **A1**

Note: Award R1 for explicit reference to the sign change of f' at each point. Award A1 for the correct nature of both stationary points. Answers without justification earn A0R0.

(c)
$x = 2.5$ **A1**

Note: The inflexion of f corresponds to the turning point (minimum) of f', which occurs at $x = 2.5$. Accept $x = \frac{5}{2}$.

Q15 — Product Rule AG then Stationary Points MEDIUM

(a)
$f'(x) = 2x \cdot e^{-x} + x^2 \cdot (-e^{-x})$ **M1**
$= e^{-x}(2x - x^2)$ **A1 AG**

Note: Award M1 for correct application of the product rule with both terms present. The final line must be shown for AG; do not accept verification by expansion alone.

(b)
Setting $f'(x) = 0$: $e^{-x}(2x - x^2) = 0$ **M1**

Since $e^{-x} > 0$ for all x: $x(2-x) = 0$, giving $x = 0$ or $x = 2$ **A1**

Note: Award A1 for both values. Award M1 for setting their $f'(x) = 0$ and factoring correctly.

Q16 — Quotient Rule Stationary Points via GDC MEDIUM

Applying the quotient rule: $f'(x) = \dfrac{3(x^2+4) - (3x+1)(2x)}{(x^2+4)^2}$ **M1**

Numerator: $3x^2 + 12 - 6x^2 - 2x = -3x^2 - 2x + 12$ **(A1)**

Setting $-3x^2 - 2x + 12 = 0$, i.e. $3x^2 + 2x - 12 = 0$, and solving using the GDC **(M1)**

$x = -2.36$ or $x = 1.69$ (3 s.f.) **A1**

Note: Full precision: $x = \dfrac{-2 \pm \sqrt{4+144}}{6} = \dfrac{-2 \pm \sqrt{148}}{6}$, giving $x = -2.360925\ldots$ and $x = 1.694258\ldots$; rounded to 3 s.f. these are -2.36 and 1.69. Award (M1) for evidence of solving $f'(x) = 0$ numerically. Award A1 for both values to 3 s.f.

Q17 — Chain Rule then Second Derivative for a Logarithm MEDIUM

(a)

$f'(x) = \dfrac{1}{x^2+3} \cdot 2x$ **M1**

$f'(x) = \dfrac{2x}{x^2+3}$ **A1**

(b)

Applying the quotient rule to $f'(x)$: $f''(x) = \dfrac{2(x^2+3) - 2x \cdot 2x}{(x^2+3)^2}$ **M1**

$f''(x) = \dfrac{2x^2 + 6 - 4x^2}{(x^2+3)^2} = \dfrac{6 - 2x^2}{(x^2+3)^2}$ **A1**

Note: Award M1 for correct structure of the quotient (or product) rule applied to $f'(x)$. Accept equivalent unsimplified forms for A1.

Q18 — Inflexion and Classification via Second Derivative MEDIUM

(a)

$f'(x) = 3x^2 - 12x + 9$, $f''(x) = 6x - 12$ **M1**

Setting $f''(x) = 0$: $6x - 12 = 0 \Rightarrow x = 2$ **A1**

Note: Candidates must find $f''(x)$ correctly and set it to zero. A sign change of f'' at $x = 2$ (from negative to positive) confirms it is a point of inflexion, but this confirmation is not required for the marks here.

(b)

$f''(1) = 6(1) - 12 = -6 < 0$ **R1**

Since $f''(1) < 0$, the stationary point at $x = 1$ is a local maximum. **A1**

Note: Award R1 for evaluating and stating the sign of $f''(1)$ explicitly. Award A1 for the correct conclusion. Answers without f'' evaluation earn R0A0.

Q19 — Chain Rule Tangent Line at a Point MEDIUM

(a)

Writing $f(x) = (4x^2 + 1)^{1/2}$ and applying the chain rule **M1**

$f'(x) = \frac{1}{2}(4x^2 + 1)^{-1/2} \cdot 8x = \frac{4x}{\sqrt{4x^2 + 1}}$ **A1**

(b)

At $x = 2$: $f(2) = \sqrt{16 + 1} = \sqrt{17}$ and $m = f'(2) = \frac{8}{\sqrt{17}}$ **(M1)**

Tangent: $y - \sqrt{17} = \frac{8}{\sqrt{17}}(x - 2)$, so $c = \sqrt{17} - \frac{16}{\sqrt{17}} = \frac{17 - 16}{\sqrt{17}} = \frac{1}{\sqrt{17}}$ **A1**

$m = \frac{8}{\sqrt{17}} \approx 1.94$, $c = \frac{1}{\sqrt{17}} \approx 0.243$ (3 s.f.)

Note: Award (M1) for substituting $x = 2$ into their $f'(x)$ and $f(x)$. Award A1 for both correct values of m and c; accept exact or 3 s.f. forms.

Q20 — Product Rule AG then Exact Stationary Point MEDIUM

(a)

$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1$ **M1**

$= \ln x + 1 - 1 = \ln x$ **A1 AG**

Note: Award M1 for applying the product rule to $x \ln x$ correctly. The -1 must be differentiated to 0; both terms must appear before simplification for AG.

(b)

Setting $f'(x) = 0$: $\ln x = 0 \Rightarrow x = 1$ **M1**

$f(1) = 1 \cdot \ln 1 - 1 = 0 - 1 = -1$, so the stationary point is $(1, -1)$ **A1**

Note: Award M1 for $\ln x = 0$ leading to $x = 1$. Award A1 for both exact coordinates stated.

Q21 — Product Rule Stationary Points and Inflection HARD

(a)

$f'(x) = 2x \cdot e^{-3x} + x^2 \cdot (-3e^{-3x})$ **M1**

$= e^{-3x}(2x - 3x^2) = xe^{-3x}(2 - 3x)$ **AG**

(b)

Setting $f'(x) = 0$: $x = 0$ or $x = \dfrac{2}{3}$ **A1**

$f''(x) = e^{-3x}(2 - 6x) + (2x - 3x^2)(-3e^{-3x}) = e^{-3x}(2 - 12x + 9x^2)$

$f''(0) = 2 > 0$, so $x = 0$ is a local minimum; $f''(\frac{2}{3}) = e^{-2}(2 - 8 + 4) = -2e^{-2} < 0$, so $x = \frac{2}{3}$ is a local maximum **A1**

Note: Both natures must be stated with f'' evidence for A1. Award A0 if only one point is addressed.

(c)

$f''(x) = 0 \Rightarrow 9x^2 - 12x + 2 = 0 \Rightarrow x = \dfrac{12 \pm \sqrt{144 - 72}}{18} = \dfrac{12 \pm 6\sqrt{2}}{18} = \dfrac{6 \pm 3\sqrt{2}}{9} = \dfrac{2 \pm \sqrt{2}}{3}$

i.e. $x = \dfrac{6 - 3\sqrt{2}}{9}$ and $x = \dfrac{6 + 3\sqrt{2}}{9}$. The sign of f'' changes at each root, confirming points of inflection. **A1**

Note: Accept either inflection x-value with sign-change evidence. A sign table or evaluating f'' either side is sufficient.

Q22 — Quotient Rule and Exact Maximum Justification HARD

(a)

$g'(x) = \dfrac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4}$ **M1**

$= \dfrac{x - 2x \ln x}{x^4} = \dfrac{1 - 2\ln x}{x^3}$ **AG**

(b)

$g'(x) = 0 \Rightarrow 1 - 2\ln x = 0 \Rightarrow \ln x = \dfrac{1}{2} \Rightarrow x = e^{1/2} = \sqrt{e}$ **M1**

$g(\sqrt{e}) = \dfrac{\ln \sqrt{e}}{(\sqrt{e})^2} = \dfrac{1/2}{e} = \dfrac{1}{2e}$; stationary point is $\left(\sqrt{e}, \dfrac{1}{2e}\right)$ **A1**

For x slightly less than $\sqrt{e}$: $\ln x < \frac{1}{2}$ so $1 - 2\ln x > 0$, i.e. $g' > 0$; for x slightly greater: $g' < 0$. The sign change $+ \to -$ confirms a local maximum. **R1**

Note: Accept $g''(\sqrt{e}) < 0$ as justification for R1. Condone $e^{0.5}$ for $\sqrt{e}$.

Q23 — Quotient Rule Stationary Point and Nature HARD

(a)

$\dfrac{dy}{dx} = \dfrac{2e^{2x}(3x - 1) - e^{2x} \cdot 3}{(3x - 1)^2} = \dfrac{e^{2x}(6x - 5)}{(3x - 1)^2}$ **M1 A1**

(b)

$\dfrac{dy}{dx} = 0 \Rightarrow 6x - 5 = 0$ (since $e^{2x} > 0$ and $(3x - 1)^2 > 0$ for $x \neq \frac{1}{3}$) **M1**

$x = \dfrac{5}{6}$ **A1**

(c)
For $x < \frac{5}{6}$: $6x - 5 < 0$ so $\frac{dy}{dx} < 0$; for $x > \frac{5}{6}$: $\frac{dy}{dx} > 0$. The sign change $- \to +$ confirms a local minimum. **R1**

Note: Award R1 only if the sign change is explicitly stated. Accept the second-derivative test if computed correctly.

Q24 — Reading the Derivative Graph Inflection and FTC Connection HARD

(a)
$f'(x)$ changes from negative to positive at $x = 1$: local minimum at $x = 1$.
$f'(x)$ changes from positive to negative at $x = 4$: local maximum at $x = 4$. **A1**

(b)
f' has its maximum at $x = 2.5$, so $f'' = 0$ (and changes sign) there; point of inflection of f at $x = 2.5$. **A1**

(c)
$f'' < 0$ (i.e. f' is decreasing) for $x \in (2.5, 5)$; f is concave down on $(2.5, 5)$. **A1**

(d)
$$f(4) - f(1) = \int_1^4 f'(x)\,dx = 4.5$$ **M1**

$f(4) = 4 + 4.5 = 8.5$ **A1**

Note: For M1 the Fundamental Theorem connection $f(4) - f(1) = \int_1^4 f'(x)\,dx$ must be seen or clearly implied. Trap: do not accept $f(4) = 4.5$ from forgetting to add $f(1)$.

Q25 — Chain Rule Twice Second Derivative and Concavity HARD

(a)
$h'(x) = \cos(x^2 + 1) \cdot 2x$ **M1**

$h''(x) = -\sin(x^2 + 1) \cdot 2x \cdot 2x + \cos(x^2 + 1) \cdot 2$ **M1**

$= 2\cos(x^2 + 1) - 4x^2 \sin(x^2 + 1)$ **AG**

(b)
$h''(0) = 2\cos(1) - 4(0)\sin(1) = 2\cos 1$ **A1**

Note: Accept $2\cos(1)$; do not accept a decimal unless the exact form is shown first.

(c)
$\cos 1 \approx 0.540 > 0$, so $h''(0) = 2\cos 1 > 0$; f is concave up at $x = 0$. **R1**

Note: R1 requires the sign conclusion and the words "concave up" (or equivalent). Award R0 if only the numerical value is stated without interpretation.

Q26 — Cubic with Local Minimum Value Giving the Constant k HARD

$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$ **M1**

Stationary points at $x = 1$ (local max) and $x = 3$ (local min), since $f''(x) = 6x - 12$: $f''(3) = 6 > 0$. **A1**

Local minimum value: $f(3) = 27 - 54 + 27 + k = k = -2$, so $k = -2$. **A1**

Local maximum at $x = 1$; $f(1) = 1 - 6 + 9 - 2 = 2$, so local maximum is $(1, 2)$. **A1**

$f''(x) = 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2$; f'' changes sign ($-$ to $+$) at $x = 2$, so point of inflection at $x = 2$. **A1**

Note: Award the third A1 only if $k = -2$ is found first. Trap: candidates may attempt to use $x = 1$ for the minimum — reject by checking $f''(1) = -6 < 0$.

Q27 — Chain Rule Twice and No-Inflection Proof HARD

(a)
Write $f(x) = (4 - x^2)^{1/2}$; $f'(x) = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x)$ **M1**

$= \dfrac{-x}{\sqrt{4 - x^2}}$ **AG**

(b)
Apply the quotient rule to $\dfrac{-x}{(4 - x^2)^{1/2}}$:

Numerator: $(-1)(4 - x^2)^{1/2} - (-x) \cdot \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = -(4 - x^2)^{1/2} - \dfrac{x^2}{(4 - x^2)^{1/2}}$ **M1**

$= \dfrac{-(4 - x^2) - x^2}{(4 - x^2)^{1/2}} = \dfrac{-4}{(4 - x^2)^{1/2}}$; dividing by $(4 - x^2)$:

$f''(x) = \dfrac{-4}{(4 - x^2)^{3/2}}$ **AG**

(c)
For all $x \in (-2, 2)$: $(4 - x^2)^{3/2} > 0$, so $f''(x) = \dfrac{-4}{(4 - x^2)^{3/2}} < 0$ always; f'' has no zero on $(-2, 2)$, hence no sign change occurs, so f has no point of inflection. **R1**

Note: R1 requires the explicit statement that f'' never equals zero AND therefore no sign change.

Q28 — Quotient Rule and GDC Stationary Points HARD

(a)
$\dfrac{dy}{dx} = \dfrac{\cos x \cdot e^x - \sin x \cdot e^x}{e^{2x}} = \dfrac{\cos x - \sin x}{e^x}$ **M1 A1**

(b)
Using the GDC to solve $\cos x - \sin x = 0$ on $[0, 2\pi]$: $\tan x = 1$, giving $x = \dfrac{\pi}{4}$ and $x = \dfrac{5\pi}{4}$ **(M1)**

$x = 0.785$ and $x = 3.93$ (3 s.f.) **A1**

Note: Both values required for A1. Accept exact $\pi/4$ and $5\pi/4$.

(c)

At $x = \pi/4$: for x slightly less, $\cos x > \sin x$ so $dy/dx > 0$; for x slightly greater, $dy/dx < 0$: local maximum. At $x = 5\pi/4$: sign change $- \to +$: local minimum. **R1**

Note: Both natures required with sign-change evidence for R1.

Q29 — Product Rule Tangent Gradient Gives the Constant p and Inflection **HARD**

(a)

$$\frac{dy}{dx} = p\,e^{-x/2} + px \cdot \left(-\tfrac{1}{2}\right) e^{-x/2} = pe^{-x/2}\left(1 - \frac{x}{2}\right)$$ **M1**

$$= \frac{pe^{-x/2}(2-x)}{2}$$ **AG**

(b)

At $x = 4$: $\dfrac{dy}{dx} = \dfrac{p\,e^{-2}(2-4)}{2} = \dfrac{-2p\,e^{-2}}{2} = -p\,e^{-2}$ **M1**

Set equal to $-e^{-2}$: $-p\,e^{-2} = -e^{-2} \Rightarrow p = 1$ **A1**

(c)

With $p = 1$: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{e^{-x/2}(2-x)}{2}\right] = \dfrac{1}{2}\left[-\dfrac{1}{2}e^{-x/2}(2-x) - e^{-x/2}\right] = \dfrac{1}{2}e^{-x/2}\left[\dfrac{x-2}{2} - 1\right] = e^{-x/2}\left(\dfrac{x-4}{4}\right)$ **M1**

$f''(x) = 0 \Rightarrow x = 4$; sign changes from $-$ to $+$, so point of inflection at $x = 4$. **A1**

Note: Trap — candidates may confuse the stationary point $x = 2$ (where $f' = 0$) with the inflection. Award A0 if $x = 2$ is stated without support.

Q30 — Rational Function Stationary Points and Inflection Decision **HARD**

Rewrite $f(x) = \dfrac{x^2 - 4x + 5}{x - 2}$; apply the quotient rule: **M1**

$f'(x) = \dfrac{(2x-4)(x-2) - (x^2-4x+5)(1)}{(x-2)^2} = \dfrac{2x^2 - 8x + 8 - x^2 + 4x - 5}{(x-2)^2} = \dfrac{x^2 - 4x + 3}{(x-2)^2} = \dfrac{(x-1)(x-3)}{(x-2)^2}$ **A1**

$f'(x) = 0 \Rightarrow x = 1$ or $x = 3$ **A1**

At $x = 1$: $f(1) = \dfrac{1-4+5}{-1} = -2$; sign of f': for $x < 1$, $(x-1) < 0$, $(x-3) < 0$, $(x-2)^2 > 0 \Rightarrow f' > 0$; for $1 < x < 2$: $(x-1) > 0$, $(x-3) < 0 \Rightarrow f' < 0$: local maximum at $(1, -2)$. **A1**

At $x = 3$: $f(3) = \dfrac{9-12+5}{1} = 2$; sign change $- \to +$: local minimum at $(3, 2)$.

$f''(x) = \dfrac{2}{(x-2)^3}$ (from differentiating); $(x-2)^3 \neq 0$ for $x \neq 2$, and $f''(x) \neq 0$ for all x in the domain, so f'' has no zero and no sign change; no point of inflection exists. **R1**

Note: R1 requires the explicit statement that f'' never equals zero on the domain and therefore no sign change. Trap: candidates may test $x = 2$ (outside the domain) as a candidate inflection — this must be rejected since f is undefined there. Award R0 if the domain exclusion is not acknowledged.