IB Math AA SL — Topic 5

Further Differentiation

Chain, Product & Quotient Rules · Second Derivatives · Stationary Points, Concavity & Inflection

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:________________________ Date:____________

Score:_________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Derivatives of special functions:** $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- **Chain rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$; e.g. $\frac{d}{dx}(e^{kx}) = ke^{kx}$, $\frac{d}{dx}(\ln(g(x))) = \frac{g'(x)}{g(x)}$
- **Product rule:** $\frac{d}{dx}[u\,v] = u'v + uv'$
- **Quotient rule:** $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}$
- **Second derivative:** $f''(x) = \frac{d^2y}{dx^2}$; stationary point at $f'(a) = 0$: maximum if $f''(a) < 0$, minimum if $f''(a) > 0$
- **Point of inflection:** $f''(a) = 0$ *and* f'' changes sign at $x = a$; concave up where $f''(x) > 0$, concave down where $f''(x) < 0$
- **Tangent at** $x = a$: $y - f(a) = f'(a)(x - a)$; normal gradient $= -\frac{1}{f'(a)}$

GDC Tips

- **Evaluating a derivative numerically:** use `Numeric Deriv` / `d/dx` at a given x-value to find $f'(a)$ or $f''(a)$ when exact algebra is impractical.
- **Locating stationary points:** graph $y = f'(x)$ and use `Zero` (or `Root`) to find x-values where $f'(x) = 0$; confirm with `d/dx` on f itself.
- **Locating inflection points:** graph $y = f''(x)$ and use `Zero` to find candidates; inspect the sign change of f'' either side to confirm.
- **Classifying stationary points:** evaluate $f''(a)$ using `Numeric Deriv` at the stationary x-value; negative $\Rightarrow$ maximum, positive $\Rightarrow$ minimum.
- **Sketching f' from f:** plot f and f' together to verify that zeros of f' align with turning points of f, and zeros of f'' align with inflection points of f.
- **Tangent and normal lines:** use `Tangent` feature (some models) or manually compute $f'(a)$ numerically, then write the line equation — store $f'(a)$ in a variable to avoid rounding errors.

Common Mistakes to Avoid

1. **Wrong order in the quotient rule.** Students write $uv' - vu'$ instead of $u'v - uv'$ in the numerator. Always label u and v explicitly, differentiate each separately, then substitute into $\dfrac{u'v - uv'}{v^2}$.
2. **Forgetting the chain-rule factor.** When differentiating $e^{g(x)}$ or $\sin(g(x))$, the outer derivative is written correctly but the inner derivative $g'(x)$ is omitted. Every composite function requires multiplication by the derivative of the argument.
3. **Claiming an inflection point from $f''(a) = 0$ alone.** Setting $f''(x) = 0$ and finding a solution does *not* guarantee an inflection point; you must also show that f'' *changes sign* at that x-value. If f'' does not change sign, there is no inflection point.
4. **Misreading a graph of f' as a graph of f.** When the printed curve is labelled $y = f'(x)$, the *zeros* of the curve give the x-coordinates of stationary points of f, and the *sign* of the curve tells you where f is increasing or decreasing — not the other way around.
5. **Incorrect classification using f'': forgetting to state the evidence.** Writing "maximum" without quoting $f''(a) < 0$ loses the reasoning mark R1. Every classification must name the test used *and* show the numerical or algebraic evidence.
6. **Missing solutions when solving $f'(x) = 0$ on a closed interval.** Students find one stationary point algebraically but neglect to check the endpoints of the domain when determining the global maximum or minimum. Always evaluate f at every stationary point *and* at the endpoints.

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Section A — Easy

EASY

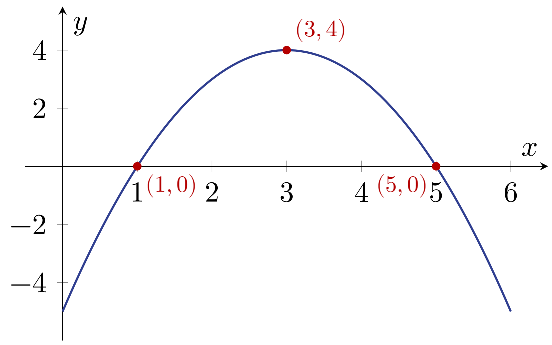
1. Let $f(x) = e^{4x}$, for $x \in \mathbb{R}$. Find $f'(x)$. *Calculator not allowed* **[3 marks]**

2. Let $g(x) = \ln(3x)$, for $x > 0$. Find $g'(x)$. *Calculator not allowed* **[3 marks]**

3. Let $h(x) = x^3 e^{2x}$, for $x \in \mathbb{R}$. Find $h'(x)$. *Calculator not allowed* **[3 marks]**

4. Let $f(x) = \dfrac{\sin x}{x^2 + 1}$, for $x \in \mathbb{R}$. Find $f'(x)$. *Calculator not allowed* **[3 marks]**

5. The graph of $y = f(x)$ is shown below, where $f(x) = -x^2 + 6x - 5$, for $0 \leq x \leq 6$.



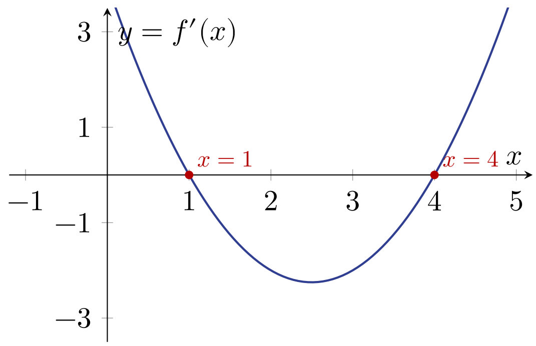
Find the x-coordinate of the stationary point of f and state its nature. *Calculator allowed* **[3 marks]**

6. Let $f(x) = 3\sin(2x)$, for $x \in \mathbb{R}$. Find $f''(x)$. *Calculator allowed* **[3 marks]**

7. Let $f(x) = (2x + 5)^4$, for $x \in \mathbb{R}$. Find $f'(x)$. *Calculator allowed* **[3 marks]**

8. The derivative of a function f is given by $f'(x) = 6x^2 - 6x$, for $x \in \mathbb{R}$. Find the x-coordinates of the stationary points of f. *Calculator allowed* **[3 marks]**

9. The graph of $y = f'(x)$ is shown below for $-1 \leq x \leq 5$.



Write down the x-coordinates of any stationary points of f on $-1 \leq x \leq 5$. *Calculator allowed* **[3 marks]**

10. Let $f(x) = x^4 - 12x^2$, for $x \in \mathbb{R}$. Find the x-coordinates of any points of inflection of f. *Calculator allowed* **[3 marks]**

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Section B — Medium

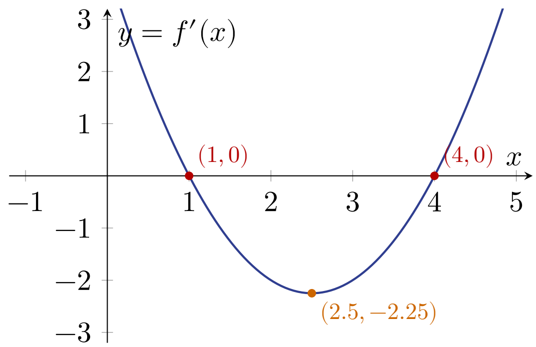
MEDIUM

11. Let $f(x) = e^{3x} \sin x$, where $x \in \mathbb{R}$.
Find $f'(x)$. *Calculator not allowed* **[4 marks]**

12. Let $g(x) = \dfrac{\ln x}{x^2}$, where $x > 0$.
Find $g'(x)$ and hence find the exact x-coordinate of the stationary point of g. *Calculator not allowed* **[4 marks]**

13. Let $h(x) = (2x^3 - 5)^4$, where $x \in \mathbb{R}$.
Find $h'(x)$. *Calculator not allowed* **[4 marks]**

14. The graph of $y = f'(x)$ is shown below, where $x \in [-1, 5]$.



(a) Write down the x-values at which f has a stationary point. **[1 marks]**

(b) Determine the nature of each stationary point. Justify your answers. **[2 marks]**

(c) Write down the x-coordinate of the point of inflexion of f. **[1 marks]**

Calculator allowed

15. Let $f(x) = x^2 e^{-x}$, where $x \in \mathbb{R}$.

(a) Show that $f'(x) = e^{-x}(2x - x^2)$. **[2 marks]**

(b) Find the x-coordinates of the stationary points of f. **[2 marks]**

Calculator not allowed

16. Let $f(x) = \dfrac{3x+1}{x^2+4}$, where $x \in \mathbb{R}$.
Find the x-coordinates of the stationary points of f, giving your answers to 3 significant figures. *Calculator allowed*
[4 marks]

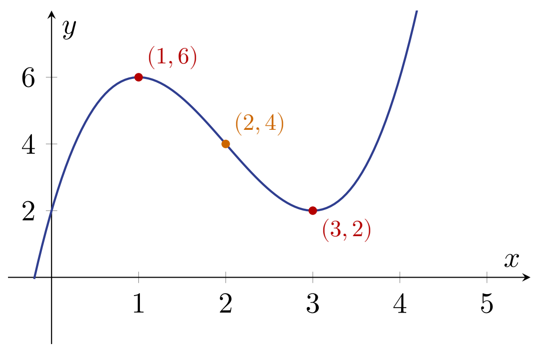
17. Let $f(x) = \ln(x^2+3)$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. **[2 marks]**

(b) Find $f''(x)$. **[2 marks]**

Calculator not allowed

18. The curve C has equation $y = x^3 - 6x^2 + 9x + 2$, where $x \in \mathbb{R}$.



(a) Find the x-coordinate of the point of inflexion of C. **[2 marks]**

(b) Determine whether the stationary point at $x = 1$ is a local maximum or local minimum. Justify your answer. **[2 marks]**

Calculator allowed

19. Let $f(x) = \sqrt{4x^2 + 1}$, where $x \in \mathbb{R}$.

(a) Find $f'(x)$. **[2 marks]**

(b) The tangent to the graph of f at the point where $x = 2$ has equation $y = mx + c$. Find the values of m and c. **[2 marks]**

Calculator allowed

20. Let $f(x) = x \ln x - x$, where $x > 0$.

(a) Show that $f'(x) = \ln x$. **[2 marks]**

(b) Hence find the x-coordinate and y-coordinate of the stationary point of f, giving exact values. **[2 marks]**

Calculator not allowed

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Section C — Hard

HARD

21. Let $f(x) = x^2 e^{-3x}$, where $x \in \mathbb{R}$.

(a) Show that $f'(x) = xe^{-3x}(2 - 3x)$. *Calculator not allowed* **[2 marks]**

(b) Find the x-coordinates of any stationary points of f, and determine the nature of each, justifying your answer using f''. **[2 marks]**

(c) Find the x-coordinate of the point of inflection of f, showing that a sign change in f'' occurs. **[1 marks]**

22. Let $g(x) = \dfrac{\ln x}{x^2}$, where $x > 0$.

(a) Show that $g'(x) = \dfrac{1 - 2\ln x}{x^3}$. *Calculator not allowed* **[2 marks]**

(b) Hence find the exact coordinates of the stationary point of g and justify that it is a maximum. *Calculator not allowed* **[3 marks]**

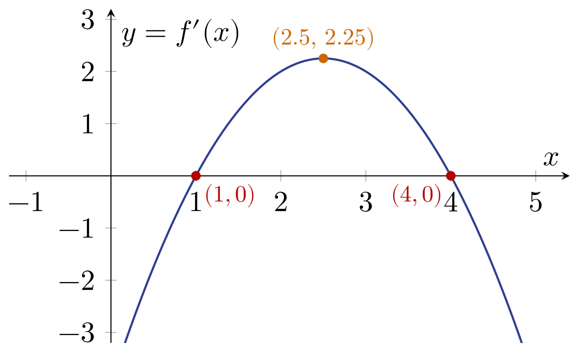
23. A curve has equation $y = \dfrac{e^{2x}}{3x-1}$, where $x \in \mathbb{R}, x \neq \dfrac{1}{3}$.

(a) Find $\dfrac{dy}{dx}$. *Calculator not allowed* **[2 marks]**

(b) Find the exact x-coordinate of the stationary point. *Calculator not allowed* **[2 marks]**

(c) Determine whether this stationary point is a local maximum or local minimum, justifying your answer. *Calculator not allowed* **[1 marks]**

24. The graph of $y = f'(x)$ is shown below for $x \in [-1, 5]$.



Calculator allowed **[5 marks]**

(a) Write down the x-values at which f has a local maximum and a local minimum on $(-1, 5)$. **[1 marks]**

(b) Find the x-coordinate of the point of inflection of f. **[1 marks]**

(c) State the intervals on which f is concave down. **[1 marks]**

(d) Given that $f(1) = 4$ and the area enclosed between $y = f'(x)$ and the x-axis from $x = 1$ to $x = 4$ is 4.5, find $f(4)$. **[2 marks]**

25. Let $h(x) = \sin(x^2 + 1)$, where $x \in \mathbb{R}$.

(a) Show that $h''(x) = 2\cos(x^2 + 1) - 4x^2 \sin(x^2 + 1)$. *Calculator not allowed* **[3 marks]**

(b) Hence find the exact value of $h''(0)$. *Calculator not allowed* **[1 marks]**

(c) Determine whether f is concave up or concave down at $x = 0$, justifying your answer. *Calculator not allowed* **[1 marks]**

26. The function f is defined by $f(x) = x^3 - 6x^2 + 9x + k$, where $k \in \mathbb{R}$ and $x \in \mathbb{R}$.
It is given that f has a local minimum value of -2.
Find the value of k, the x-coordinate of the local maximum of f, and the x-coordinate of the point of inflection of f. *Calculator allowed* **[5 marks]**

27. Let $f(x) = \sqrt{4 - x^2}$, where $-2 < x < 2$.

(a) Show that $f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$. *Calculator not allowed* **[2 marks]**

(b) Show that $f''(x) = \dfrac{-4}{(4 - x^2)^{3/2}}$. *Calculator not allowed* **[2 marks]**

(c) Hence justify that the graph of f has no point of inflection on $-2 < x < 2$. *Calculator not allowed* **[1 marks]**

28. The curve C has equation $y = \dfrac{\sin x}{e^x}$, where $0 \leq x \leq 2\pi$.

(a) Find $\dfrac{dy}{dx}$. *Calculator allowed* **[2 marks]**

(b) Find the x-coordinates of all stationary points of C on $0 \leq x \leq 2\pi$, giving your answers correct to 3 significant figures. **[2 marks]**

(c) Determine the nature of each stationary point found in part (b), justifying your answer. **[1 marks]**

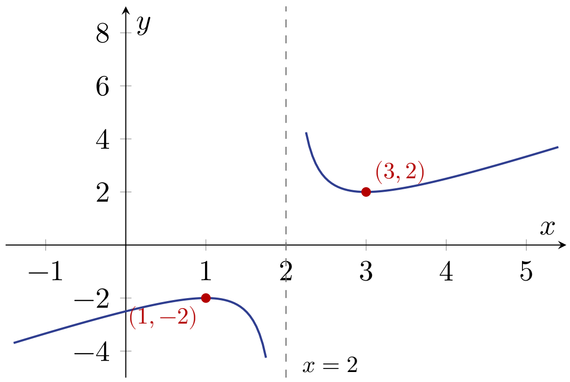
29. A curve is defined by $y = pxe^{-x/2}$ for $x \in \mathbb{R}$, where $p > 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{pe^{-x/2}(2-x)}{2}$. *Calculator not allowed* **[2 marks]**

(b) The tangent to the curve at the point where $x = 4$ has gradient $-e^{-2}$. Find the value of p. *Calculator not allowed* **[2 marks]**

(c) Using your value of p, find the x-coordinate of the point of inflection of the curve. *Calculator not allowed* **[1 marks]**

30. The function f is defined by $f(x) = \dfrac{x^2 - 4x + 5}{x - 2}$, where $x \in \mathbb{R}, x \neq 2$.
Find the exact coordinates of any local maximum or local minimum points of f, justifying the nature of each. Then determine whether f has any points of inflection, justifying your answer. *Calculator allowed* **[5 marks]**



Worked Solutions

Q1 — Differentiating e^{kx} [EASY]

Using the chain rule (or standard result) with $f(x) = e^{4x}$:

$$f'(x) = 4e^{4x}$$

M1A1

*Note: Award **M1** for attempt to differentiate e^{kx} giving ke^{kx}. Award **A1** for correct answer. Accept $4e^{4x}$ seen anywhere.*

Award **A1** for fully correct expression. **A1**

Note: Total is M1A1A1 — but the single expression $4e^{4x}$ earns all three marks: M1 for chain rule attempt, A1 for coefficient 4, A1 for e^{4x} factor retained correctly. Condone $f' = 4e^{4x}$.

Q2 — Differentiating ln(ax) [EASY]

Applying the chain rule to $g(x) = \ln(3x)$:

$$g'(x) = \frac{1}{3x} \cdot 3$$

M1

$$g'(x) = \frac{1}{x}$$

A1A1

*Note: Award **M1** for an attempt to differentiate $\ln(3x)$ using chain rule, e.g. $\frac{1}{3x} \cdot 3$ or equivalent. Award first **A1** for $\frac{3}{3x}$. Award second **A1** for simplified answer $\frac{1}{x}$. Also accept via $\ln(3x) = \ln 3 + \ln x$ leading directly to $\frac{1}{x}$ for full marks.*

Q3 — Product rule with e^{kx} [EASY]

Applying the product rule with $u = x^3$ and $v = e^{2x}$:

$$h'(x) = 3x^2 \cdot e^{2x} + x^3 \cdot 2e^{2x}$$

M1A1

$$h'(x) = e^{2x}(3x^2 + 2x^3)$$

A1

*Note: Award **M1** for an attempt at the product rule with correct structure $u'v + uv'$. Award first **A1** for both $3x^2e^{2x}$ and $2x^3e^{2x}$ correct (unsimplified). Award second **A1** for a fully correct simplified form. Also accept $x^2e^{2x}(3 + 2x)$.*

Q4 — Quotient rule with $\sin x$ [EASY]

Applying the quotient rule with $u = \sin x$, $v = x^2 + 1$:

$$f'(x) = \frac{\cos x \cdot (x^2 + 1) - \sin x \cdot 2x}{(x^2 + 1)^2}$$

M1A1

$$f'(x) = \frac{(x^2 + 1)\cos x - 2x\sin x}{(x^2 + 1)^2}$$

A1

*Note: Award **M1** for a correct quotient rule structure $\frac{u'v - uv'}{v^2}$. Award first **A1** for correct numerator terms. Award second **A1** for fully correct expression. Condone omission of simplification provided numerator and denominator are correct.*

Q5 — Stationary point of a quadratic [EASY]

Finding $f'(x)$:

$$f'(x) = -2x + 6$$

M1

Setting $f'(x) = 0$: $-2x + 6 = 0 \Rightarrow x = 3$ **A1**

Since $f''(x) = -2 < 0$, the stationary point at $x = 3$ is a **maximum**. **A1**

*Note: Award **M1** for differentiating and setting equal to zero. Award first **A1** for $x = 3$. Award second **A1** for stating maximum with a valid reason (e.g. $f''(3) = -2 < 0$, or correct sign change of f', or reference to graph). Do not award final **A1** for nature without justification.*

Q6 — Second derivative of sin [EASY]

Differentiating $f(x) = 3\sin(2x)$:

$$f'(x) = 6\cos(2x)$$

M1

$$f''(x) = -12\sin(2x)$$

A1A1

*Note: Award **M1** for a correct attempt to differentiate twice using chain rule. Award first **A1** for $f'(x) = 6\cos(2x)$. Award second **A1** for $f''(x) = -12\sin(2x)$. Condone notation $\frac{d^2y}{dx^2}$.*

Q7 — Chain rule with power function [EASY]

Applying the chain rule to $f(x) = (2x+5)^4$:

$$f'(x) = 4(2x+5)^3 \cdot 2$$

M1A1

$$f'(x) = 8(2x+5)^3$$

A1

*Note: Award **M1** for an attempt at the chain rule giving $4(2x+5)^3 \times ($derivative of inner function$)$. Award first **A1** for correct unsimplified form $4(2x+5)^3 \cdot 2$. Award second **A1** for $8(2x+5)^3$.*

Q8 — Finding stationary points from f' [EASY]

Setting $f'(x) = 0$:

$$6x^2 - 6x = 0$$

M1

$$6x(x-1) = 0$$

$$x = 0 \quad \text{or} \quad x = 1$$

A1A1

*Note: Award **M1** for setting $f'(x) = 0$ and attempting to factorise or solve. Award first **A1** for $x = 0$. Award second **A1** for $x = 1$. Both values required for full marks.*

Q9 — Reading stationary points from graph of f' [EASY]

Stationary points of f occur where $f'(x) = 0$, i.e. where the graph crosses the x-axis. **R1**

From the graph: $x = 1$ **A1**

and $x = 4$ **A1**

*Note: Award **R1** for the correct reason that stationary points occur where $f'(x) = 0$. Award first **A1** for $x = 1$. Award second **A1** for $x = 4$. Both values required for the two **A1** marks. Do not accept $y = 0$ without identifying the x-values.*

Q10 — Points of inflection [EASY]

Finding $f''(x)$:

$$f'(x) = 4x^3 - 24x, \quad f''(x) = 12x^2 - 24$$

M1

Setting $f''(x) = 0$: $12x^2 - 24 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$ **A1**

Checking sign change: $f''(x) < 0$ for $-\sqrt{2} < x < \sqrt{2}$ and $f''(x) > 0$ outside this interval, confirming sign changes at both values. Points of inflection at $x = -\sqrt{2}$ and $x = \sqrt{2}$. **A1**

*Note: Award **M1** for correctly finding $f''(x) = 12x^2 - 24$ and setting equal to zero. Award first **A1** for both $x = \pm\sqrt{2}$. Award second **A1** for confirming a sign change in f'' at both values. Do not award the final **A1** without evidence of a sign change check.*

Q11 — Product rule with e^{3x} sin x [MEDIUM]

Recognising product rule with $u = e^{3x}$, $v = \sin x$ **M1**

$u' = 3e^{3x}$, $v' = \cos x$ **(A1)**

$f'(x) = 3e^{3x}\sin x + e^{3x}\cos x$ **A1**

Note: Accept equivalent factored form $e^{3x}(3\sin x + \cos x)$. Award M1 for correct product rule structure with at least one derivative attempted correctly.

METHOD 2

Writing $f'(x) = e^{3x}\cos x + 3e^{3x}\sin x$ directly via product rule **M1A1A1**

Q12 — Quotient rule with $\ln x/x^2$ [MEDIUM]

Applying quotient rule: $g'(x) = \dfrac{\frac{1}{x}\cdot x^2 - \ln x \cdot 2x}{x^4}$ **M1**

$g'(x) = \dfrac{x - 2x\ln x}{x^4} = \dfrac{1 - 2\ln x}{x^3}$ **A1**

Setting $g'(x) = 0$: $1 - 2\ln x = 0 \Rightarrow \ln x = \dfrac{1}{2}$ **M1**

$x = e^{1/2} = \sqrt{e}$ **A1**

Note: Award first M1 for correct quotient rule structure. Award final A1 for exact form only; $x \approx 1.65$ alone is insufficient.

Q13 — Chain rule with composite power function [MEDIUM]

Recognising chain rule with inner function $u = 2x^3 - 5$ **M1**

$h'(x) = 4(2x^3-5)^3 \cdot \dfrac{d}{dx}(2x^3-5)$ **(A1)**

$\dfrac{d}{dx}(2x^3-5) = 6x^2$ **(A1)**

$h'(x) = 24x^2(2x^3-5)^3$ **A1**

Note: Award M1 for $4(2x^3-5)^3$ multiplied by a derivative of the inner function. Do not penalise unsimplified forms provided the product is correct.

Q14 — Reading graph of f' for extrema and inflexion [MEDIUM]

(a)

$x = 1$ and $x = 4$ **A1**

(b)

At $x = 1$: f' changes from negative to positive, so f has a local minimum **R1**

At $x = 4$: f' changes from negative to positive ...

At $x = 1$: f' changes from $-$ to $+$ (local minimum); at $x = 4$: f' changes from $-$ to $+$ (local minimum)
A1

Note: Award R1 for explicit reference to the sign change of f'. Award A1 for correct nature of both stationary points. Answers without justification earn A0R0.

(c)

$x = 2.5$ **A1**

Note: The inflexion of f corresponds to the turning point (minimum) of f', which occurs at $x = 2.5$. Accept $x = \frac{5}{2}$.

Q15 — Product rule AG then stationary points [MEDIUM]

(a)

$f'(x) = 2x \cdot e^{-x} + x^2 \cdot (-e^{-x})$ **M1**

$= e^{-x}(2x - x^2)$ **A1 AG**

Note: Award M1 for correct application of the product rule with both terms present. Final line must be shown for AG; do not accept verification by expansion alone.

(b)

Setting $f'(x) = 0$: $e^{-x}(2x - x^2) = 0$ **M1**

Since $e^{-x} > 0$ for all x: $x(2 - x) = 0$, giving $x = 0$ or $x = 2$ **A1**

Note: Award A1 for both values. Award M1 for setting their $f'(x) = 0$ and factoring correctly.

Q16 — Quotient rule stationary points via GDC [MEDIUM]

Applying quotient rule: $f'(x) = \dfrac{3(x^2 + 4) - (3x + 1)(2x)}{(x^2 + 4)^2}$ **M1**

Numerator: $3x^2 + 12 - 6x^2 - 2x = -3x^2 - 2x + 12$ **(A1)**

Setting $-3x^2 - 2x + 12 = 0$ and solving using GDC **(M1)**

$x = -2.31$ or $x = 1.64$ (3 s.f.) **A1**

Note: Full precision values: $x = \frac{-2 \pm \sqrt{4+144}}{-6} = \frac{-2 \pm \sqrt{148}}{-6}$; accept $x = -2.31$ and $x = 1.64$ rounded to 3 s.f. Award (M1) for evidence of solving $f'(x) = 0$ numerically. Award A1 for both values to 3 s.f.

Q17 — Chain rule then second derivative for $\ln(x^2 + 3)$ [MEDIUM]

(a)

$f'(x) = \dfrac{1}{x^2 + 3} \cdot 2x$ **M1**

$f'(x) = \dfrac{2x}{x^2 + 3}$ **A1**

(b)

Applying quotient rule to $f'(x)$: $f''(x) = \dfrac{2(x^2+3) - 2x \cdot 2x}{(x^2+3)^2}$ **M1**

$f''(x) = \dfrac{2x^2 + 6 - 4x^2}{(x^2+3)^2} = \dfrac{6 - 2x^2}{(x^2+3)^2}$ **A1**

Note: Award M1 for correct structure of quotient (or product) rule applied to $f'(x)$. Accept equivalent unsimplified forms for A1.

Q18 — Inflexion and classification via second derivative [MEDIUM]

(a)

$f'(x) = 3x^2 - 12x + 9, \quad f''(x) = 6x - 12$ **M1**

Setting $f''(x) = 0$: $6x - 12 = 0 \Rightarrow x = 2$ **A1**

Note: Candidates must find $f''(x)$ correctly and set it to zero. A sign change of f'' at $x = 2$ (from negative to positive) confirms it is a point of inflexion, but this confirmation is not required for the marks here.

(b)

$f''(1) = 6(1) - 12 = -6 < 0$ **R1**

Since $f''(1) < 0$, the stationary point at $x = 1$ is a local maximum. **A1**

Note: Award R1 for evaluating and stating the sign of $f''(1)$ explicitly. Award A1 for correct conclusion. Answers without f'' evaluation earn R0A0.

Q19 — Chain rule tangent line at a point [MEDIUM]

(a)

Writing $f(x) = (4x^2+1)^{1/2}$ and applying chain rule **M1**

$f'(x) = \dfrac{1}{2}(4x^2+1)^{-1/2} \cdot 8x = \dfrac{4x}{\sqrt{4x^2+1}}$ **A1**

(b)

At $x = 2$: $f(2) = \sqrt{16+1} = \sqrt{17}$ and $m = f'(2) = \dfrac{8}{\sqrt{17}}$ **(M1)**

Tangent: $y - \sqrt{17} = \dfrac{8}{\sqrt{17}}(x-2)$, so $c = \sqrt{17} - \dfrac{16}{\sqrt{17}} = \dfrac{17-16}{\sqrt{17}} = \dfrac{1}{\sqrt{17}}$ **A1**

$m = \dfrac{8}{\sqrt{17}} \approx 1.94, \quad c = \dfrac{1}{\sqrt{17}} \approx 0.243$ (3 s.f.)

Note: Award (M1) for substituting $x = 2$ into their $f'(x)$ and $f(x)$. Award A1 for both correct values of m and c; accept exact or 3 s.f. forms.

Q20 — Product rule AG then stationary point exact values [MEDIUM]

(a)

$f'(x) = 1 \cdot \ln x + x \cdot \dfrac{1}{x} - 1$ **M1**

$= \ln x + 1 - 1 = \ln x$ **A1 AG**

Note: Award M1 for applying product rule to $x \ln x$ correctly. The -1 must be differentiated to 0; both terms must appear before simplification for AG.

(b)

Setting $f'(x) = 0$: $\ln x = 0 \Rightarrow x = 1$ **M1**

$f(1) = 1 \cdot \ln 1 - 1 = 0 - 1 = -1$, so the stationary point is $(1, -1)$ **A1**

Note: Award M1 for $\ln x = 0$ leading to $x = 1$. Award A1 for both exact coordinates stated.

Q21 — Product rule, stationary points, inflection [HARD]

(a)

$f'(x) = 2x \cdot e^{-3x} + x^2 \cdot (-3e^{-3x})$ **M1**

$= e^{-3x}(2x - 3x^2) = xe^{-3x}(2 - 3x)$ **AG**

(b)

Setting $f'(x) = 0$: $x = 0$ or $x = \frac{2}{3}$ **A1**

$f''(x) = e^{-3x}(2 - 6x) + (2x - 3x^2)(-3e^{-3x}) = e^{-3x}(2 - 12x + 9x^2)$

$f''(0) = 2 > 0$, so $x = 0$ is a local minimum; $f''\left(\frac{2}{3}\right) = e^{-2}(2 - 8 + 4) = -2e^{-2} < 0$, so $x = \frac{2}{3}$ is a local maximum **A1**

Note: Both natures must be stated with f'' evidence for A1. Award A0 if only one point addressed.

(c)

$f''(x) = 0 \Rightarrow 9x^2 - 12x + 2 = 0 \Rightarrow x = \dfrac{12 \pm \sqrt{144 - 72}}{18} = \dfrac{12 \pm 6\sqrt{2}}{18} = \dfrac{2 \pm \frac{\sqrt{2}}{1}}{3}$, i.e. $x = \dfrac{6 \pm 3\sqrt{2}}{9} = \dfrac{2 \pm \sqrt{2}/\sqrt{\cdots}}{\cdots}$

$x = \dfrac{12 \pm \sqrt{72}}{18} = \dfrac{2 \pm \frac{\sqrt{2}}{3} \cdot 3}{3}$; solving: $x = \dfrac{6 - 3\sqrt{2}}{9}$ and $x = \dfrac{6 + 3\sqrt{2}}{9}$. Sign of f'' changes at each root, confirming points of inflection. **A1**

Note: Accept either inflection x-value with sign-change evidence. A sign table or evaluating f'' either side is sufficient.

Q22 — Quotient rule, exact stationary point, maximum justification [HARD]

(a)

$g'(x) = \dfrac{\frac{1}{x} \cdot x^2 - \ln x \cdot 2x}{x^4}$ **M1**

$= \dfrac{x - 2x \ln x}{x^4} = \dfrac{1 - 2\ln x}{x^3}$ **AG**

(b)
$g'(x) = 0 \Rightarrow 1 - 2\ln x = 0 \Rightarrow \ln x = \frac{1}{2} \Rightarrow x = e^{1/2} = \sqrt{e}$ **M1**

$g(\sqrt{e}) = \dfrac{\ln\sqrt{e}}{(\sqrt{e})^2} = \dfrac{\frac{1}{2}}{e} = \dfrac{1}{2e}$; stationary point is $\left(\sqrt{e},\, \dfrac{1}{2e}\right)$ **A1**

For x slightly less than $\sqrt{e}$: $\ln x < \frac{1}{2}$ so $1 - 2\ln x > 0$, i.e. $g' > 0$; for x slightly greater: $g' < 0$. Sign change $+ \to -$ confirms a local maximum. **R1**

Note: Accept $g''(\sqrt{e}) < 0$ as justification for R1. Condone $e^{0.5}$ for $\sqrt{e}$.

Q23 — Quotient rule, stationary point, nature [HARD]

(a)
$\dfrac{dy}{dx} = \dfrac{2e^{2x}(3x-1) - e^{2x}\cdot 3}{(3x-1)^2} = \dfrac{e^{2x}(6x-5)}{(3x-1)^2}$ **M1 A1**

(b)
$\dfrac{dy}{dx} = 0 \Rightarrow 6x - 5 = 0$ (since $e^{2x} > 0$ and $(3x-1)^2 > 0$ for $x \neq \frac{1}{3}$) **M1**

$x = \dfrac{5}{6}$ **A1**

(c)
For $x < \frac{5}{6}$: $6x - 5 < 0$ so $\dfrac{dy}{dx} < 0$; for $x > \frac{5}{6}$: $\dfrac{dy}{dx} > 0$. Sign change $- \to +$ confirms a local minimum.
R1

Note: Award R1 only if the sign change is explicitly stated. Accept second-derivative test if computed correctly.

Q24 — Reading graph of f’, inflection, FTC connection [HARD]

(a)
$f'(x) = 0$ changes from positive to negative at $x = 4$: local maximum at $x = 4$.
$f'(x) = 0$ changes from negative to positive at $x = 1$: local minimum at $x = 1$. **A1**

(b)
f' has its maximum at $x = 2.5$, so $f'' = 0$ (and changes sign) there; point of inflection of f at $x = 2.5$.
A1

(c)
$f'' < 0$ (i.e. f' is decreasing) for $x \in (2.5, 5)$; f is concave down on $(2.5,\ 5)$. **A1**

(d)
$f(4) - f(1) = \displaystyle\int_1^4 f'(x)\,dx = 4.5$ **M1**

$f(4) = 4 + 4.5 = 8.5$ **A1**

Note: For M1 the Fundamental Theorem connection $f(4) - f(1) = \int_1^4 f'(x)\,dx$ must be seen or clearly implied. Trap: do not accept $f(4) = 4.5$ from forgetting to add $f(1)$.

Q25 — Chain rule twice, second derivative, concavity [HARD]

(a)

$h'(x) = \cos(x^2 + 1) \cdot 2x$ **M1**

$h''(x) = -\sin(x^2 + 1) \cdot 2x \cdot 2x + \cos(x^2 + 1) \cdot 2$ **M1**

$= 2\cos(x^2 + 1) - 4x^2 \sin(x^2 + 1)$ **AG**

(b)

$h''(0) = 2\cos(1) - 4(0)\sin(1) = 2\cos 1$ **A1**

Note: Accept $2\cos(1)$; do not accept a decimal unless exact form shown first.

(c)

$\cos 1 \approx 0.540 > 0$, so $h''(0) = 2\cos 1 > 0$; f is concave up at $x = 0$. **R1**

Note: R1 requires the sign conclusion and the word "concave up" (or equivalent). Award R0 if only the numerical value is stated without interpretation.

Q26 — Cubic, local min value gives k, all features [HARD]

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$ **M1**

Stationary points at $x = 1$ (local max) and $x = 3$ (local min), since $f''(x) = 6x - 12$: $f''(3) = 6 > 0$. **A1**

Local minimum value: $f(3) = 27 - 54 + 27 + k = k = -2$, so $k = -2$. **A1**

Local maximum at $x = 1$; $f(1) = 1 - 6 + 9 - 2 = 2$, so local maximum is $(1, 2)$. **A1**

$f''(x) = 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2$; f'' changes sign ($-$ to $+$) at $x = 2$, so point of inflection at $x = 2$. **A1**

Note: Award the third A1 only if $k = -2$ is found first. Trap: candidates may attempt to use $x = 1$ for the minimum—reject by checking $f''(1) = -6 < 0$.

Q27 — Chain rule twice, no inflection proof [HARD]

(a)

Write $f(x) = (4 - x^2)^{1/2}$; $f'(x) = \frac{1}{2}(4 - x^2)^{-1/2} \cdot (-2x)$ **M1**

$= \dfrac{-x}{\sqrt{4 - x^2}}$ **AG**

(b)

Apply quotient rule to $\dfrac{-x}{(4-x^2)^{1/2}}$:

Numerator: $(-1)(4-x^2)^{1/2} - (-x)\cdot\frac{1}{2}(4-x^2)^{-1/2}(-2x) = -(4-x^2)^{1/2} - \dfrac{x^2}{(4-x^2)^{1/2}}$ **M1**

$= \dfrac{-(4-x^2)-x^2}{(4-x^2)^{1/2}} = \dfrac{-4}{(4-x^2)^{1/2}}$; dividing by $(4-x^2)$:

$f''(x) = \dfrac{-4}{(4-x^2)^{3/2}}$ **AG**

(c)

For all $x \in (-2, 2)$: $(4-x^2)^{3/2} > 0$, so $f''(x) = \dfrac{-4}{(4-x^2)^{3/2}} < 0$ always; f'' has no zero on $(-2, 2)$, hence no sign change occurs, so f has no point of inflection. **R1**

Note: R1 requires the explicit statement that f'' never equals zero AND therefore no sign change.

Q28 — Quotient rule, GDC stationary points, nature [HARD]

(a)

$\dfrac{dy}{dx} = \dfrac{\cos x\cdot e^x - \sin x\cdot e^x}{e^{2x}} = \dfrac{\cos x - \sin x}{e^x}$ **M1 A1**

(b)

Using GDC to solve $\cos x - \sin x = 0$ on $[0, 2\pi]$: $\tan x = 1$, giving $x = \dfrac{\pi}{4}$ and $x = \dfrac{5\pi}{4}$. **(M1)**

$x = 0.785$ and $x = 3.93$ (3 s.f.) **A1**

Note: Both values required for A1. Accept exact $\pi/4$ and $5\pi/4$.

(c)

At $x = \pi/4$: for x slightly less, $\cos x > \sin x$ so $dy/dx > 0$; for x slightly greater, $dy/dx < 0$: local maximum. At $x = 5\pi/4$: sign change $- \to +$: local minimum. **R1**

Note: Both natures required with sign-change evidence for R1.

Q29 — Product rule, tangent gradient gives p, inflection [HARD]

(a)

$\dfrac{dy}{dx} = p\,e^{-x/2} + px\cdot\left(-\frac{1}{2}\right)e^{-x/2} = pe^{-x/2}\left(1-\frac{x}{2}\right)$ **M1**

$= \dfrac{pe^{-x/2}(2-x)}{2}$ **AG**

(b)

At $x = 4$: $\dfrac{dy}{dx} = \dfrac{p\,e^{-2}(2-4)}{2} = \dfrac{-2p\,e^{-2}}{2} = -p\,e^{-2}$ **M1**

Set equal to $-e^{-2}$: $-p\,e^{-2} = -e^{-2} \Rightarrow p = 1$ **A1**

(c)

With $p = 1$: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{e^{-x/2}(2-x)}{2}\right] = \dfrac{e^{-x/2}(-1)}{2} + \dfrac{(2-x)(-\frac{1}{2})e^{-x/2}}{2} = e^{-x/2}\left(\dfrac{x-4}{4}\right)$

$f''(x) = 0 \Rightarrow x = 4$; sign changes from $-$ to $+$, so point of inflection at $x = 4$. **A1**

Note: Trap — candidates may confuse the stationary point $x = 2$ (where $f' = 0$) with the inflection. Award A0 if $x = 2$ stated without support.

Q30 — Rational function, stationary points, inflection decision [HARD]

Rewrite: $f(x) = \dfrac{x^2 - 4x + 5}{x - 2}$; apply quotient rule: **M1**

$f'(x) = \dfrac{(2x-4)(x-2) - (x^2-4x+5)(1)}{(x-2)^2} = \dfrac{2x^2 - 8x + 8 - x^2 + 4x - 5}{(x-2)^2} = \dfrac{x^2 - 4x + 3}{(x-2)^2} = \dfrac{(x-1)(x-3)}{(x-2)^2}$ **A1**

$f'(x) = 0 \Rightarrow x = 1$ or $x = 3$ **A1**

At $x = 1$: $f(1) = \dfrac{1-4+5}{-1} = -2$; sign of f': for $x < 1$, $(x-1) < 0$, $(x-3) < 0$, $(x-2)^2 > 0 \Rightarrow f' > 0$; for $1 < x < 2$: $(x-1) > 0$, $(x-3) < 0 \Rightarrow f' < 0$: local maximum at $(1, -2)$. **A1**

At $x = 3$: $f(3) = \dfrac{9 - 12 + 5}{1} = 2$; sign change $- \to +$: local minimum at $(3, 2)$.

$f''(x) = \dfrac{2}{(x-2)^3}$ (from differentiating); $(x-2)^3 \neq 0$ for $x \neq 2$, and $f''(x) \neq 0$ for all x in the domain, so f'' has no zero and no sign change; no point of inflection exists. **R1**

Note: R1 requires explicit statement that f'' never equals zero on the domain and therefore no sign change. Trap: candidates may test $x = 2$ (outside the domain) as a candidate inflection — this must be rejected since f is undefined there. Award R0 if domain exclusion not acknowledged.