IB Math AA SL — Topic 5

Further Integration

Special Functions · Reverse Chain Rule & Substitution · Signed Areas · Areas Between Curves

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Scan me for help

Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

Reverse chain rule (linear inner): $\displaystyle\int f(ax+b)\,dx = \tfrac{1}{a}F(ax+b)+C,$ where $F' = f$

$$\int e^{kx}\,dx = \tfrac{1}{k}e^{kx}+C \qquad \int \frac{1}{ax+b}\,dx = \tfrac{1}{a}\ln|ax+b|+C$$

$$\int \sin(ax+b)\,dx = -\tfrac{1}{a}\cos(ax+b)+C \qquad \int \cos(ax+b)\,dx = \tfrac{1}{a}\sin(ax+b)+C$$

Substitution: if $u = g(x)$ then $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du;$ change limits $x = a \Rightarrow u = g(a),\ x = b \Rightarrow u = g(b)$

Definite integral: $\displaystyle\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

Area under a curve (below axis): split at roots $f(c) = 0;$ Area $= \displaystyle\int_a^c f(x)\,dx + \left|\int_c^b f(x)\,dx\right|$

Area between two curves: $A = \displaystyle\int_a^b [f(x) - g(x)]\,dx$ where $f(x) \geq g(x)$ on $[a, b]$

GDC Tips

1. **Definite integral (area or signed):** use Calculus $\to \int dx$ (TI-Nspire) or Math $\to$ 9:fnInt((TI-84); the GDC returns the signed value, so check whether the curve dips below the axis.
2. **Finding intersections for area limits:** graph both functions and use Analyze Graph $\to$ Intersection (Nspire) or 2nd CALC $\to$ 5:intersect (TI-84).

3. **Splitting a below-axis integral:** find roots with Analyze Graph $\rightarrow$ Zero or 2nd CALC $\rightarrow$ 2:zero; evaluate each piece separately and add absolute values.
4. **Checking a reverse-chain or substitution result:** differentiate your answer using Calculus $\rightarrow d/dx$ and confirm it matches the original integrand.
5. **Storing intermediate values:** store exact intersection coordinates or area pieces with STO$\rightarrow$ before combining, to avoid rounding errors when summing.

Section A — Easy EASY (10 questions × 3 marks = 30 marks)

Scan me for help

1. EASY No Calculator [3 marks]

Find $\int 5e^{2x}\,dx$, where $x \in \mathbb{R}$.

2. EASY No Calculator [3 marks]

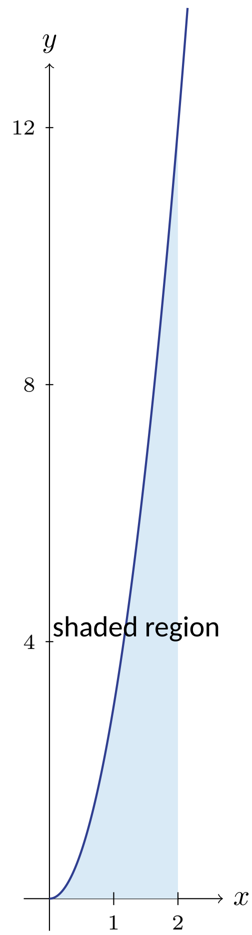
Find $\int \frac{3}{x}\,dx$, where $x > 0$.

3. EASY No Calculator [3 marks]

Find $\int 4\cos(2x)\,dx$, where $x \in \mathbb{R}$.

4. EASY Calculator **[3 marks]**

The diagram below shows the graph of $f(x) = 3x^2$ for $0 \leq x \leq 2$.



Find the area of the shaded region.

5. EASY Calculator **[3 marks]**

Find $\displaystyle\int_1^4 \frac{2}{\sqrt{x}}\,dx$.

6. EASY No Calculator **[3 marks]**

Let $f(x) = \sin(3x)$ for $x \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

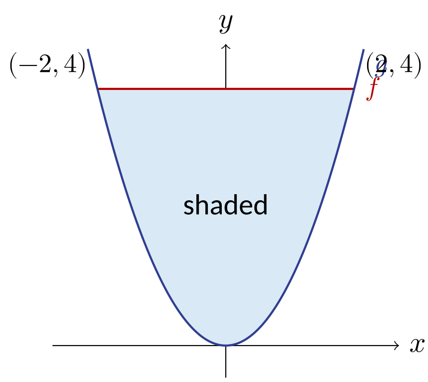
(b) Hence find $\displaystyle\int 6\cos(3x)\,dx$. [2]

7. EASY Calculator **[3 marks]**

Find $\displaystyle\int_0^3 (2x+1)\,dx$ using the substitution $u = 2x+1$.

8. EASY Calculator **[3 marks]**

The diagram below shows the graphs of $f(x) = 4$ and $g(x) = x^2$ for $-2 \leq x \leq 2$.



Find the area of the shaded region enclosed between the two curves.

9. EASY Calculator **[3 marks]**

Find $\displaystyle\int_0^{\pi} \sin x \, dx$.

10. EASY Calculator **[3 marks]**

Find $\displaystyle\int 8(2x+3)^3 \, dx$, where $x \in \mathbb{R}$. Express your answer in the form $a(2x+3)^4 + C$, where $a \in \mathbb{Q}$ and C is a constant.

Section B — Medium MEDIUM (10 questions × 4 marks = 40 marks)

Scan me for help

11. MEDIUM No Calculator **[4 marks]**

Find $\displaystyle\int \frac{4}{3x-1}\,dx$, where $x > \frac{1}{3}$.

12. MEDIUM No Calculator **[4 marks]**

Let $f(x) = \sin(5x-2)$ for $x \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

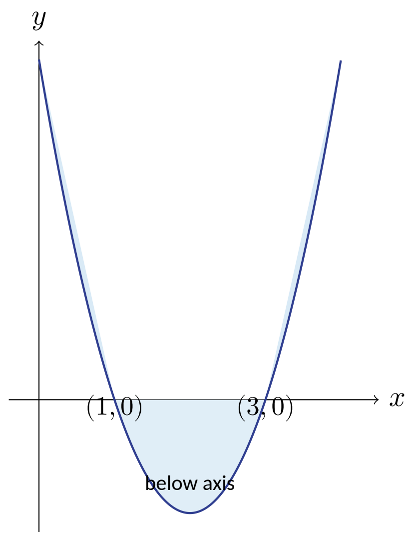
(b) Hence find $\displaystyle\int 10\cos(5x-2)\,dx$. [3]

13. MEDIUM No Calculator **[4 marks]**

Find $\displaystyle\int_0^1 x\,e^{x^2+1}\,dx$ using the substitution $u = x^2+1$. Give your answer in the form $\dfrac{a}{2}(e^b - e)$, where $a, b \in \mathbb{Z}^+$.

14. MEDIUM Calculator **[4 marks]**

The graph of $y = f(x)$ where $f(x) = 3x^2 - 12x + 9$ is shown below for $0 \leq x \leq 4$.



Find the total area of the shaded regions between the curve and the x-axis for $0 \leq x \leq 4$.

15. MEDIUM No Calculator **[4 marks]**

Find $\displaystyle\int_1^4 \frac{3}{\sqrt{x}}\,dx$.

16. MEDIUM No Calculator **[4 marks]**

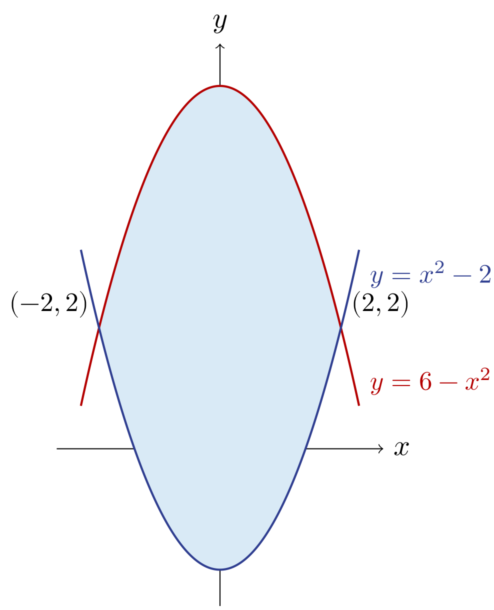
Let $f(x) = e^{3x}(2x+1)$ for $x \in \mathbb{R}$.

(a) Show that $f'(x) = e^{3x}(6x+5)$. [2]

(b) Hence find $\displaystyle\int e^{3x}(6x+5)\,dx$. [2]

17. MEDIUM Calculator **[4 marks]**

The region enclosed by the curves $y = x^2 - 2$ and $y = 6 - x^2$ is shown below.



Find the area of the enclosed region.

18. MEDIUM No Calculator **[4 marks]**

Let $g(x) = \cos^2 x$ for $x \in \mathbb{R}$. Using the identity $\cos 2x = 2\cos^2 x - 1$, find $\int_0^{\pi/4} \cos^2 x \, dx$. Give your answer as an exact value.

19. MEDIUM Calculator **[4 marks]**

Find the value of $c > 1$ such that $\int_1^c \frac{2}{x} \, dx = 6$.

20. MEDIUM Calculator **[4 marks]**

The velocity of a particle, in m s^{-1}, at time t seconds ($t \geq 0$) is given by $v(t) = 4\sin(2t) - 2$ for $0 \leq t \leq \pi$.

(a) Find the values of t at which $v(t) = 0$ for $0 \leq t \leq \pi$. [2]

(b) Find the total distance travelled by the particle for $0 \leq t \leq \pi$. [2]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

Scan me for help

21. **HARD** **No Calculator** **[5 marks]**

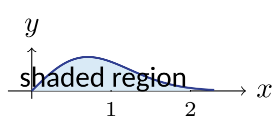
Let $f(x) = \ln(3x^2 + 1)$, where $x \in \mathbb{R}$.

(a) Show that $f'(x) = \dfrac{6x}{3x^2+1}$. [1]

(b) Hence find $\displaystyle\int_0^2 \frac{12x}{3x^2+1}\,dx$, giving your answer in the form $\ln k$ where $k \in \mathbb{Z}^+$. [4]

22. **HARD** **No Calculator** **[5 marks]**

The diagram below shows part of the curve $y = x\,e^{-x^2}$ for $x \geq 0$.



Find the exact area of the shaded region bounded by the curve, the x-axis, $x = 0$ and $x = 2$.

23. HARD No Calculator **[5 marks]**

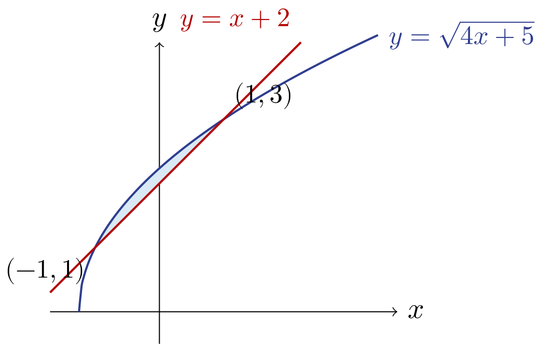
Let $g(x) = \dfrac{\cos x}{2 + \sin x}$, where $x \in \mathbb{R}$.

(a) Find $g'(x)$. [2]

(b) Hence find the exact value of $\displaystyle\int_0^{\pi/2} \frac{\cos x}{2 + \sin x}\, dx$. [3]

24. HARD Calculator **[5 marks]**

Consider the curve $y = \sqrt{4x + 5}$ and the line $y = x + 2$.



Find the exact area of the region enclosed between the curve and the line.

25. HARD Calculator **[5 marks]**

The velocity of a particle moving along a straight line, for $t \geq 0$ seconds, is given by

$$v(t) = 2\cos\left(\frac{\pi t}{3}\right) - 1, \qquad t \in [0, 6].$$

Find the total distance travelled by the particle in the interval $0 \leq t \leq 6$.

26. HARD No Calculator **[5 marks]**

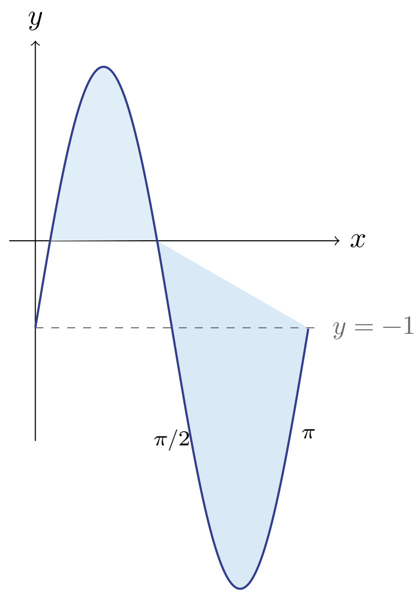
Use the substitution $u = x^2 + 4$ to find the exact value of

$$\int_0^2 \frac{x^3}{x^2+4}\,dx.$$

Express your answer in the form $a + b\ln c$, where $a, b \in \mathbb{Q}$ and $c > 0$.

27. HARD Calculator **[5 marks]**

The diagram shows the curve $y = 3\sin(2x) - 1$ for $0 \le x \le \pi$.



Find the total area of the regions enclosed between the curve $y = 3\sin(2x) - 1$ and the x-axis for $0 \le x \le \pi$.

28. HARD Calculator **[5 marks]**

Find the value of the positive constant c such that

$$\int_1^c \left(3x^2 - \frac{2}{x}\right) dx = 10.$$

29. HARD No Calculator **[5 marks]**

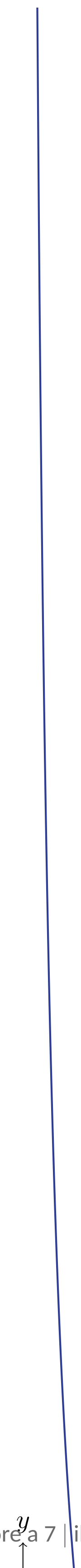
The curve C has equation $y = e^{2x} - 6e^{x} + 5$, where $x \in \mathbb{R}$.

(a) Show that C crosses the x-axis at $x = 0$ and $x = \ln 5$. [2]

(b) Find the exact area of the region enclosed between C and the x-axis. [3]

30. **HARD** **Calculator** **[5 marks]**

Consider the two curves $y = \dfrac{3}{x}$ and $y = 7 - e^{x}$, where $x > 0$.


y

These curves intersect at two points. Find the area of the region enclosed between the two curves, correct to 3 significant figures.

Worked Solutions

M = method mark A = accuracy mark R = reasoning mark AG = answer given FT = follow-through

Q1 — Integrate ekx EASY

Attempt to integrate $5e^{2x}$, applying the reverse chain rule **M1**

$\int 5e^{2x}\,dx = \frac{5}{2}e^{2x} + C$ **A1A1**

Note: Award A1 for e^{2x} (or equivalent), A1 for coefficient $\frac{5}{2}$. Accept $5/2\,e^{2x} + C$ or $2.5e^{2x} + C$.

Q2 — Integrate 1 over x EASY

Attempt to integrate $\frac{3}{x}$ **M1**

$\int \frac{3}{x}\,dx = 3\ln x + C$ **A1A1**

Note: Award A1 for $\ln x$ (modulus not required since $x > 0$ is stated), A1 for coefficient 3 with $+C$. Accept 3 ln x + C or 3 ln|x| + C.

Q3 — Integrate cos(kx) EASY

Attempt to integrate $4\cos(2x)$, applying the reverse chain rule **M1**

$\int 4\cos(2x)\,dx = \frac{4}{2}\sin(2x) + C = 2\sin(2x) + C$ **A1A1**

Note: Award A1 for $\sin(2x)$, A1 for coefficient 2 with $+C$. Accept 2 sin(2x) + C.

Q4 — Definite Integral Area Under Curve EASY

Attempt to integrate $3x^2$ from 0 to 2 **M1**

$\int_0^2 3x^2\,dx = [x^3]_0^2$ **(A1)**

$= 8 - 0 = 8$ **A1**

Note: Answer 8 (square units). Accept any equivalent exact form.

Q5 — Definite Integral Fractional Power EASY

Write $\frac{2}{\sqrt{x}} = 2x^{-1/2}$ and attempt to integrate **M1**

$$\int_1^4 2x^{-1/2}\,dx = [4x^{1/2}]_1^4$$ **(A1)**

$= 4(2) - 4(1) = 8 - 4 = 4$ **A1**

Note: Accept 4 (exact).

Q6 — Differentiate then Integrate EASY

(a) $f'(x) = 3\cos(3x)$ **A1**

(b) Recognise $6\cos(3x) = 2 \times 3\cos(3x) = 2f'(x)$ **M1**

$$\int 6\cos(3x)\,dx = 2\sin(3x) + C$$ **A1**

Note: Accept f'(x) = 3 cos(3x); hence the integral equals 2 sin(3x) + C.

Q7 — Integration by Substitution EASY

Let $u = 2x + 1$, so $\dfrac{du}{dx} = 2$, i.e. $dx = \dfrac{du}{2}$ **M1**

Change limits: $x = 0 \Rightarrow u = 1$; $x = 3 \Rightarrow u = 7$ **(A1)**

$$\int_1^7 u \cdot \frac{du}{2} = \left[\frac{u^2}{4}\right]_1^7 = \frac{49}{4} - \frac{1}{4} = \frac{48}{4} = 12$$ **A1**

Note: $dx = du/2$ used correctly; limits must be changed or the working must return to x before substituting.

Q8 — Area Between Two Curves EASY

Attempt area $= \displaystyle\int_{-2}^{2} (4 - x^2)\,dx$, top minus bottom **M1**

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$ **(A1)**

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{16}{3} + \frac{16}{3} = \frac{32}{3}$$ **A1**

Note: Accept $32/3$ or 10.7 (3 s.f.). Condone symmetry argument $2\int_0^2 (4 - x^2)\,dx$ leading to the same answer.

Q9 — Definite Integral of sin x **EASY**

Attempt to integrate $\sin x$ over $[0, \pi]$ **M1**

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}$$ **(A1)**

$= -\cos\pi - (-\cos 0) = -(-1) + 1 = 2$ **A1**

Note: Since $\sin x \geq 0$ on $[0, \pi]$, no sign issue arises. Answer 2 (exact).

Q10 — Integrate Linear Composite **EASY**

Attempt to integrate $8(2x+3)^3$ by the reverse chain rule **M1**

$$\int 8(2x+3)^3 \, dx = \frac{8}{4 \times 2}(2x+3)^4 + C = (2x+3)^4 + C$$ **A1**

$= 1 \cdot (2x+3)^4 + C$, so $a = 1$ **A1**

Note: Award M1 for $(2x+3)^4$ seen with an attempt to divide by both 4 and 2. Final A1 requires the answer written in the stated form with $a = 1$ identified.

Q11 — Integrating 1 over (ax+b) **MEDIUM**

Recognise the integral as a multiple of $\ln|3x-1|$ **M1**

$$\int \frac{4}{3x-1} \, dx = \tfrac{4}{3}\ln(3x-1) + C$$ **A1A1A1**

Note: Award A1 for $\ln(3x-1)$ (or $\ln|3x-1|$), A1 for coefficient $4/3$, A1 for $+C$. Since $x > 1/3$, accept $\ln(3x-1)$ without modulus.

Q12 — Differentiate then Integrate **MEDIUM**

(a) $f'(x) = 5\cos(5x-2)$ **A1**

(b) Observe that $10\cos(5x-2) = 2 \times f'(x)$ **M1**

$$\int 10\cos(5x-2) \, dx = 2\sin(5x-2) + C$$ **A1A1**

Note: Accept f'(x) = 5 cos(5x-2); hence the integral equals 2 sin(5x-2) + C.

Q13 — Integration by Substitution Definite **MEDIUM**

Let $u = x^2 + 1 \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow x\,dx = \dfrac{du}{2}$ **M1**

Change limits: $x = 0 \Rightarrow u = 1$; $x = 1 \Rightarrow u = 2$ **(A1)**

$$\int_1^2 e^u \cdot \frac{du}{2} = \frac{1}{2}[e^u]_1^2 = \frac{1}{2}(e^2 - e)$$ **A1**

Written in the required form: $\frac{1}{2}(e^2 - e)$, so $a = 1, b = 2$ **A1**

Note: Do not award the final A1 if the student returns to x before evaluating.

Q14 — Negative Integral Total Area MEDIUM

Roots of f: $3x^2 - 12x + 9 = 0 \Rightarrow x = 1$ and $x = 3$ (seen anywhere) **(A1)**

Split the integral at the roots and use the GDC **M1**

$$\text{Area} = \int_0^1 f(x)\,dx + \left|\int_1^3 f(x)\,dx\right| + \int_3^4 f(x)\,dx$$

$$= [x^3 - 6x^2 + 9x]_0^1 + \left|[x^3 - 6x^2 + 9x]_1^3\right| + [x^3 - 6x^2 + 9x]_3^4$$

$$= 4 + |-4| + 4 = 12$$ **A1A1**

Note: A bare signed integral $\int_0^4 f\,dx = 4$ quoted as the area receives M0.

Q15 — Exact Definite Integral Fractional Power MEDIUM

Write $\frac{3}{\sqrt{x}} = 3x^{-1/2}$ and integrate **M1**

$$\int_1^4 3x^{-1/2}\,dx = \left[3 \cdot \frac{x^{1/2}}{1/2}\right]_1^4 = [6\sqrt{x}]_1^4$$ **A1**

$$= 6\sqrt{4} - 6\sqrt{1} = 12 - 6 = 6$$ **A1A1**

Note: Award A1 for antiderivative $6x^{1/2}$; A1 for correct substitution of limits; A1 for 6.

Q16 — Differentiate then Integrate Product Rule MEDIUM

(a) Apply the product rule to $f(x) = e^{3x}(2x + 1)$: **M1**

$f'(x) = 3e^{3x}(2x + 1) + e^{3x}(2) = e^{3x}(6x + 3 + 2) = e^{3x}(6x + 5)$ **AG A1**

(b) Since $\frac{d}{dx}\left[e^{3x}(2x + 1)\right] = e^{3x}(6x + 5)$: **M1**

$$\int e^{3x}(6x + 5)\,dx = e^{3x}(2x + 1) + C$$ **A1**

Note: In (a), award M1 for correct application of the product rule with both terms present; the final line is AG and carries no mark.

Q17 — Area Between Two Curves MEDIUM

Intersections: $x^2 - 2 = 6 - x^2 \Rightarrow 2x^2 = 8 \Rightarrow x = \pm 2$ (seen anywhere) **(A1)**

Top minus bottom: $(6 - x^2) - (x^2 - 2) = 8 - 2x^2$, integrate from -2 to 2 **M1**

$$\int_{-2}^{2} (8 - 2x^2)\,dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$$

$$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right) = \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$$ **A1A1**

Note: Accept $64/3$ or 21.3 (3 s.f.).

Q18 — Exact Definite Integral using Trig Identity MEDIUM

From the identity: $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ **M1**

$$\int_0^{\pi/4} \cos^2 x\,dx = \int_0^{\pi/4} \frac{1 + \cos 2x}{2}\,dx = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi/4}$$ **A1**

$$= \left(\tfrac{\pi}{8} + \tfrac{\sin(\pi/2)}{4}\right) - (0 + 0) = \pi/8 + 1/4$$ **A1A1**

Note: Do not accept a decimal approximation for the final mark.

Q19 — Unknown Upper Limit from Given Area MEDIUM

Evaluate the integral: **M1**

$$\int_1^{c} \frac{2}{x}\,dx = [2\ln|x|]_1^c = 2\ln c - 2\ln 1 = 2\ln c$$ **A1**

Set equal to 6 and solve: $2\ln c = 6 \Rightarrow \ln c = 3 \Rightarrow c = e^3$ **(M1)**

(GDC: $c = 20.1$) **A1**

Note: Final A1 for $c = e^3$ (exact) or 20.1 (3 s.f.); both acceptable.

Q20 — Velocity Roots then Total Distance MEDIUM

(a) Solve $4\sin(2t) - 2 = 0 \Rightarrow \sin(2t) = \frac{1}{2}$ **M1**

$$2t = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow t = \frac{\pi}{12},\ t = \frac{5\pi}{12}$$ **A1**

(b) Antiderivative: $F(t) = -2t - 2\cos(2t)$ (verified $F' = v$). Split at the roots from (a) and evaluate each piece: **M1**

Distance $= \left|F(\tfrac{\pi}{12}) - F(0)\right| + \left|F(\tfrac{5\pi}{12}) - F(\tfrac{\pi}{12})\right| + \left|F(\pi) - F(\tfrac{5\pi}{12})\right|$

$= 0.2557\ldots + 1.3697\ldots + 7.3972\ldots \approx 9.02$ m **A1**

Note: Award M1 for splitting the integral at the roots from (a) (or equivalent GDC total-distance method); A1 for 9.02 m (3 s.f.). Full precision: $9.0226\ldots$ m. A bare $\int_0^\pi v\,dt$ quoted as distance receives M0.

Q21 — Differentiate then Integrate ln Composite HARD

(a) Differentiating $\ln(3x^2+1)$ using the chain rule: $f'(x) = \dfrac{6x}{3x^2+1}$ **AG A1**

(b) Recognise $\dfrac{12x}{3x^2+1} = 2 \cdot \dfrac{6x}{3x^2+1} = 2f'(x)$ **M1**

$\displaystyle\int_0^2 \frac{12x}{3x^2+1}\,dx = [2\ln(3x^2+1)]_0^2$ **A1**

$= 2\ln 13 - 2\ln 1 = 2\ln 13$ **A1**

Express in the required form: $k = 13^2 = 169$, so answer $= \ln 169$ **A1**

Note: The required form is $\ln k$, so $\ln 169$ is required for the final A1.

Q22 — Substitution with Area Reverse Chain HARD

Let $u = -x^2$, so $\dfrac{du}{dx} = -2x$, giving $x\,dx = -\frac{1}{2}\,du$ **M1**

Change limits: $x = 0 \Rightarrow u = 0$; $x = 2 \Rightarrow u = -4$ **A1**

$\displaystyle\int_0^2 x\,e^{-x^2}\,dx = \int_0^{-4} e^u\left(-\tfrac{1}{2}\right)du = \frac{1}{2}\int_{-4}^0 e^u\,du$ **M1**

$= \frac{1}{2}[e^u]_{-4}^0 = \frac{1}{2}(e^0 - e^{-4}) = \frac{1}{2}(1 - e^{-4})$ **A1**

Since $y \geq 0$ on $[0, 2]$, the area equals $\frac{1}{2}(1 - e^{-4})$ **A1**

Note: Trap: limits must be changed or the student must explicitly return to x.

Q23 — Quotient Rule then Hence Integrate HARD

(a) Attempt quotient rule: $g'(x) = \dfrac{-\sin x(2+\sin x) - \cos x \cdot \cos x}{(2+\sin x)^2}$ **M1**

$= \dfrac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} = \dfrac{-2\sin x - 1}{(2+\sin x)^2}$ **A1**

(b) Recognise $\dfrac{\cos x}{2+\sin x}$ is the derivative of $\ln(2+\sin x)$ by inspection (reverse chain rule) **M1**

$\displaystyle\int_0^{\pi/2} \frac{\cos x}{2+\sin x}\,dx = [\ln(2+\sin x)]_0^{\pi/2}$ **A1**

$= \ln(2+1) - \ln(2+0) = \ln 3 - \ln 2 = \ln(3/2)$ **A1**

Note: The "hence" instruction means part (a) must be seen but does not have to be used explicitly; direct reverse-chain recognition is accepted.

Q24 — Area Between Curve and Line HARD

Find intersections: $\sqrt{4x+5} = x+2$ **M1**

Squaring: $4x+5 = (x+2)^2 = x^2+4x+4 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ **(A1)**

Check both in the original (unsquared) equation: $x=-1$: $\sqrt{1} = 1 = (-1)+2$ ✓; $x = 1$: $\sqrt{9} = 3 = 1+2$ ✓. Both genuine (no extraneous root). **A1**

At $x=0$: curve $= \sqrt{5} \approx 2.236$, line $= 2$, so the curve is on top on $(-1, 1)$ **M1**

Area $= \displaystyle\int_{-1}^{1} \left[\sqrt{4x+5} - (x+2)\right] dx = \left[\tfrac{1}{6}(4x+5)^{3/2} - \tfrac{x^2}{2} - 2x\right]_{-1}^{1}$

$= \left(\tfrac{1}{6}(9)^{3/2} - \tfrac{1}{2} - 2\right) - \left(\tfrac{1}{6}(1)^{3/2} - \tfrac{1}{2} + 2\right) = \left(\tfrac{27}{6} - \tfrac{5}{2}\right) - \left(\tfrac{1}{6} + \tfrac{3}{2}\right)$

Area $= 1/3$ (exact) **A1**

Note: Exact area $= 1/3 \approx 0.333$ (3 s.f.). Both intersection x-coordinates (-1 and 1) must be verified against the unsquared equation, since squaring can introduce extraneous roots.

Q25 — Distance from Velocity with Sign Split HARD

Find roots of $v(t) = 0$ on $[0, 6]$: $2\cos(\pi t/3) = 1 \Rightarrow \cos(\pi t/3) = \frac{1}{2}$ **M1**

$\pi t/3 = \pi/3$ or $5\pi/3$, giving $t = 1$ and $t = 5$ **A1**

Check signs: $v(0) = 2(1) - 1 = 1 > 0$; $v(3) = 2(-1) - 1 = -3 < 0$; $v(6) = 2(1) - 1 = 1 > 0$ **(M1)**

Antiderivative: $F(t) = \dfrac{6}{\pi}\sin(\pi t/3) - t$ (verified $F' = v$).

Total distance $= |F(1) - F(0)| + |F(5) - F(1)| + |F(6) - F(5)|$

$= 0.6540\ldots + 7.3080\ldots + 0.6540\ldots \approx 8.62$ m **A1**

Note: Accept answers in the range $[8.61, 8.63]$. Full precision: $8.6159\ldots$ m. Trap: using $\int_0^6 v\,dt$ (displacement) rather than splitting at the roots $t = 1$ and $t = 5$ is penalised by withholding the final A1.

Q26 — Substitution with Limit Change Exact Form HARD

$u = x^2 + 4 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,du$; also $x^2 = u - 4$ **M1**

Change limits: $x = 0 \Rightarrow u = 4$; $x = 2 \Rightarrow u = 8$ **A1**

$\displaystyle\int_0^2 \frac{x^3}{x^2+4}\,dx = \int_4^8 \frac{u-4}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_4^8 \left(1 - \frac{4}{u}\right) du$ **M1**

$= \frac{1}{2}[u - 4\ln|u|]_4^8 = \frac{1}{2}\left[(8 - 4\ln 8) - (4 - 4\ln 4)\right]$ **A1**

$= \frac{1}{2}(4 - 4\ln 8 + 4\ln 4) = 2 - 2\ln 8 + 2\ln 4 = 2 + 2\ln\frac{4}{8} = 2 - 2ln2$ **A1**

Note: So $a = 2$, $b = -2$, $c = 2$. Accept equivalent forms such as $2 + \ln\frac{1}{4}$.

Q27 — Total Area with Below-Axis Split HARD

Find zeros of $3\sin(2x) - 1 = 0$ on $[0, \pi]$: $\sin(2x) = \frac{1}{3}$ **M1**

$2x = \arcsin\frac{1}{3}$ or $\pi - \arcsin\frac{1}{3}$, giving $x_1 = \frac{1}{2}\arcsin\frac{1}{3} \approx 0.170$ and $x_2 = \frac{1}{2}\left(\pi - \arcsin\frac{1}{3}\right) \approx 1.40$ **A1**

Sign check: curve positive on (x_1, x_2), negative on $[0, x_1) \cup (x_2, \pi]$ **(M1)**

Antiderivative: $F(x) = -\frac{3}{2}\cos(2x) - x$ (verified $F' = f$).

Total area $= |F(x_1) - F(0)| + |F(x_2) - F(x_1)| + |F(\pi) - F(x_2)|$

$= 0.0841\ldots + 1.5975\ldots + 4.6549\ldots \approx 6.34$ **A1**

Note: Accept answers in the range $[6.33, 6.35]$. Full precision roots: $x_1 = 0.169918\ldots$, $x_2 = 1.400878\ldots$; full precision area $6.3365\ldots$. Penalise by withholding the final A1 if the student computes $\int_0^\pi (3\sin 2x - 1)\,dx$ without splitting. The split at the roots is mandatory.

Q28 — Unknown Upper Limit from Given Area HARD

Integrate: $\displaystyle\int_1^c \left(3x^2 - \frac{2}{x}\right) dx = [x^3 - 2\ln|x|]_1^c$ **M1**

$= (c^3 - 2\ln c) - (1 - 2\ln 1) = c^3 - 2\ln c - 1$ **A1**

Set equal to 10: $c^3 - 2\ln c - 1 = 10 \Rightarrow c^3 - 2\ln c = 11$ **M1**

Using the GDC to solve numerically: $c \approx 2.33$ **(A1)**

Reject $c \leq 0$ (logarithm undefined); accept only the positive solution: $c \approx 2.33$ **A1**

Note: Full precision from GDC: $c = 2.3327\ldots \approx 2.33$ (3 s.f.). The root-rejection reasoning (R mark embedded in the final A1) must be present.

Q29 — Show AG then Exact Enclosed Area HARD

(a) At $x = 0$: $y = e^0 - 6e^0 + 5 = 1 - 6 + 5 = 0$ ✓ **A1**

At $x = \ln 5$: $y = e^{2\ln 5} - 6e^{\ln 5} + 5 = 25 - 30 + 5 = 0$ ✓ **A1**

(b) Curve is below the x-axis on $(0, \ln 5)$ since $y(1) = e^2 - 6e + 5 \approx -6.92 < 0$ **M1**

$$\text{Area} = -\int_0^{\ln 5} (e^{2x} - 6e^x + 5)\,dx = -\left[\frac{e^{2x}}{2} - 6e^x + 5x\right]_0^{\ln 5}$$

$= -\left[\left(\frac{25}{2} - 30 + 5\ln 5\right) - \left(\frac{1}{2} - 6\right)\right] = -[-12 + 5\ln 5]$ **A1**

Area $= 12 - 5ln5$ **A1**

Note: $12 - 5\ln 5 \approx 3.95 > 0$, confirmed. AG firewall in part (a): exact algebra required, not a decimal check.

Q30 — Area Between Two Curves GDC Intersections **HARD**

Find intersections of $\dfrac{3}{x} = 7 - e^x$ for $x > 0$ using the GDC **M1**

GDC gives: $x_1 \approx 0.574$ and $x_2 \approx 1.64$ **A1**

Check which is on top: at $x = 1.11$, $\dfrac{3}{1.11} \approx 2.70$ and $7 - e^{1.11} \approx 3.97$, so $7 - e^x$ is above **(M1)**

Area $= \displaystyle\int_{0.574}^{1.64} \left[(7 - e^x) - \frac{3}{x}\right] dx = [7x - e^x - 3\ln x]_{0.574}^{1.64}$ evaluated using the GDC **(A1)**

Area ≈ 0.933 **A1**

Note: Accept answers in the range $[0.931, 0.935]$. Full GDC precision: $x_1 = 0.57425\ldots$, $x_2 = 1.64384\ldots$, Area $= 0.93275\ldots \approx 0.933$ (3 s.f.). Top-minus-bottom must be exhibited.

Common Mistakes to Avoid

1. **Forgetting the $\frac{1}{a}$ factor in the reverse chain rule.** When integrating $f(ax+b)$, students write $F(ax+b)+C$ instead of $\frac{1}{a}F(ax+b)+C$. This error is especially common with e^{kx}, $\sin(ax)$, and $\cos(ax)$.

2. **Failing to change limits in a definite substitution.** When $u = g(x)$ is used in a definite integral, the limits must be converted to u-values via $u = g(a)$ and $u = g(b)$. Leaving the original x-limits on the u-integral gives a wrong numerical answer.

3. **Quoting a signed integral as an area below the axis.** If the curve lies below the x-axis on part of the interval, $\int_a^b f(x)\,dx$ is negative or reduced in magnitude. The area is $|\int f(x)\,dx|$, not the raw signed value. Always locate roots, split the integral, and take absolute values of negative pieces before summing.

4. **Integrating $\frac{1}{x}$ as a power instead of a logarithm.** Students write $\int \frac{1}{x}\,dx = \frac{x^0}{0}$, which is undefined. The correct result is $\ln|x| + C$.

5. **Reversing the top-minus-bottom order between two curves.** The area between f and g requires $\int_a^b [f(x) - g(x)]\,dx$ with $f(x) \geq g(x)$ on $[a, b]$. Writing $g - f$ gives a negative result. Always verify which function is on top by testing a point inside the interval before setting up the integral.

6. **Squaring introduces extraneous roots.** When solving $\sqrt{\ldots} =$ line by squaring, always check each resulting root against the *original* (unsquared) equation — squaring can also silently eliminate genuine solutions if the domain is not respected, or create fake ones.