IB Math AA SL — Topic 5

Further Integration

Special Functions · Reverse Chain Rule & Substitution · Signed Areas · Areas Between Curves

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Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Reverse chain rule (linear inner):** $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C$, where $F' = f$
- **Key antiderivatives:** $\int e^{kx}\,dx = \frac{1}{k}e^{kx}+C$; $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|+C$; $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+C$; $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+C$
- **Substitution:** if $u = g(x)$ then $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$; for definite integrals change limits: $x = a \Rightarrow u = g(a)$, $x = b \Rightarrow u = g(b)$
- **Definite integral:** $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$
- **Area under a curve (below axis):** split at roots $x = c$ where $f(c) = 0$; Area $= \int_a^c f(x)\,dx + \left|\int_c^b f(x)\,dx\right|$
- **Area between two curves:** $A = \int_a^b [f(x) - g(x)]\,dx$ where $f(x) \geq g(x)$ on $[a, b]$; intersections give a and b
- **Area between a curve and a line:** identify top function minus bottom function on each sub-interval; confirm sign before integrating

GDC Tips

- **Definite integral (area or signed):** use `Calculus` $\rightarrow$ $\int$ `dx` (TI-Nspire) or `Math` $\rightarrow$ `9:fnInt(` (TI-84); enter the integrand, variable, lower limit, upper limit; the GDC returns the *signed* value, so check whether the curve dips below the axis.
- **Finding intersections for area limits:** graph both functions and use `Analyze Graph` $\rightarrow$ `Intersection` (Nspire) or `2nd CALC` $\rightarrow$ `5:intersect` (TI-84) to obtain exact or decimal crossing points to use as limits.
- **Splitting a below-axis integral:** find roots with `Analyze Graph` $\rightarrow$ `Zero` or `2nd CALC` $\rightarrow$ `2:zero`; evaluate each piece separately and add absolute values — never trust a single `fnInt` over the whole interval.
- **Checking a reverse-chain or substitution result:** differentiate your answer using `Calculus` $\rightarrow$ `d/dx` and confirm it matches the original integrand; catches missing $\frac{1}{a}$ factors instantly.
- **Storing intermediate values:** store exact intersection coordinates or area pieces with `STO`$\rightarrow$ before combining; avoids rounding errors when summing two area pieces.
- **Verifying an exact answer:** after obtaining a surd/pi answer by hand, evaluate it numerically and compare with the GDC's `fnInt` result; values should agree to at least 6 significant figures.

Common Mistakes to Avoid

1. **Forgetting the $\frac{1}{a}$ factor in the reverse chain rule.** When integrating $f(ax+b)$, students write $F(ax+b)+C$ instead of $\frac{1}{a}F(ax+b)+C$. This error is especially common with e^{kx}, $\sin(ax)$, and $\cos(ax)$. Always divide by the coefficient of x inside the function.

2. **Failing to change limits in a definite substitution.** When $u = g(x)$ is used in a definite integral, the limits must be converted to u-values via $u = g(a)$ and $u = g(b)$. Leaving the original x-limits on the u-integral gives a wrong numerical answer; returning to x first is the safe alternative but limits must still be applied correctly.

3. **Quoting a signed integral as an area below the axis.** If the curve lies below the x-axis on part of the interval, $\int_a^b f(x)\,dx$ is negative or reduced in magnitude. The area is $\left|\int_c^b f(x)\,dx\right|$, not the raw signed value. Always locate roots, split the integral, and take absolute values of negative pieces before summing.

4. **Integrating $\frac{1}{x}$ as a power instead of a logarithm.** Students write $\int \frac{1}{x}\,dx = \frac{x^0}{0}$, which is undefined. The correct result is $\ln|x| + C$. The modulus must be retained until the final numerical substitution confirms the argument is positive.

5. **Reversing the top-minus-bottom order between two curves.** The area between f and g requires $\int_a^b [f(x) - g(x)]\,dx$ with $f(x) \geq g(x)$ on $[a, b]$. Writing $g - f$ gives a negative result. Always verify which function is on top by testing a point inside the interval before setting up the integral.

6. **Using a single integral over the whole interval when one curve overtakes the other.** If f and g intersect inside (a, b), the top/bottom relationship switches and the integral must be split at the interior intersection point, with the correct order applied to each piece separately. A single unsplit integral will undercount the total enclosed area.

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Section A — Easy

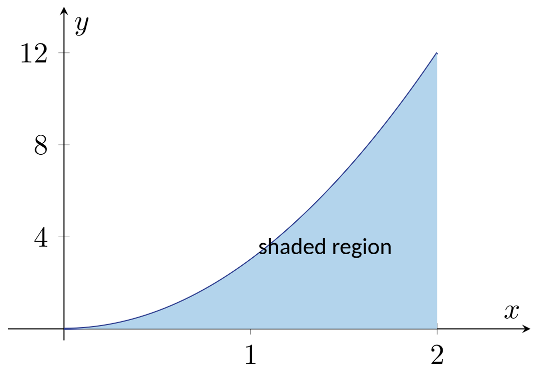
EASY

1. Find $\int 5e^{2x}\,dx$, where $x \in \mathbb{R}$. *Calculator not allowed* **[3 marks]**

2. Find $\int \frac{3}{x}\,dx$, where $x > 0$. *Calculator not allowed* **[3 marks]**

3. Find $\int 4\cos(2x)\,dx$, where $x \in \mathbb{R}$. *Calculator not allowed* **[3 marks]**

4. The diagram below shows the graph of $f(x) = 3x^2$ for $0 \leq x \leq 2$.



Find the area of the shaded region. *Calculator allowed* **[3 marks]**

5. Find $\displaystyle\int_1^4 \frac{2}{\sqrt{x}}\, dx$. *Calculator allowed* **[3 marks]**

6. Let $f(x) = \sin(3x)$ for $x \in \mathbb{R}$.

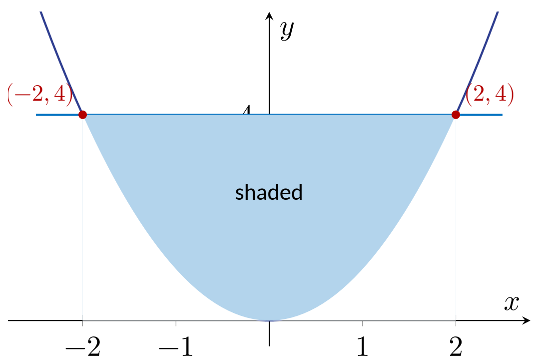
(a) Find $f'(x)$. **[1 marks]**

(b) Hence find $\displaystyle\int 6\cos(3x)\,dx$. **[2 marks]**

Calculator not allowed **[3 marks]**

7. Find $\displaystyle\int_0^3 (2x+1)\,dx$ using the substitution $u = 2x+1$. *Calculator allowed* **[3 marks]**

8. The diagram below shows the graphs of $f(x) = 4$ and $g(x) = x^2$ for $-2 \leq x \leq 2$.



Find the area of the shaded region enclosed between the two curves. *Calculator allowed* **[3 marks]**

9. Find $\int_0^{\pi} \sin x \, dx$. *Calculator allowed* **[3 marks]**

10. Find $\int 8(2x+3)^3 \, dx$, where $x \in \mathbb{R}$. Express your answer in the form $a(2x+3)^4 + C$, where $a \in \mathbb{Q}$ and C is a constant. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. Find $\displaystyle\int \frac{4}{3x-1}\,dx$, where $x > \frac{1}{3}$. *Calculator not allowed* **[4 marks]**

12. Let $f(x) = \sin(5x-2)$ for $x \in \mathbb{R}$.

(a) Find $f'(x)$. **[1 marks]**

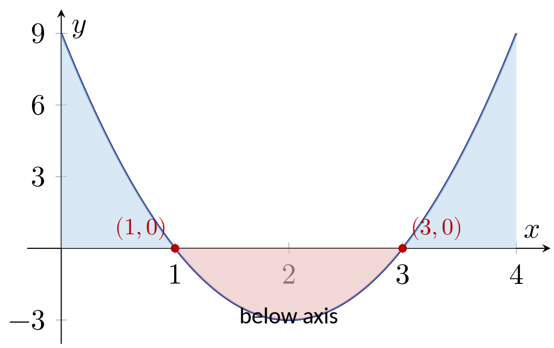
(b) Hence find $\displaystyle\int 10\cos(5x-2)\,dx$. **[3 marks]**

Calculator not allowed **[4 marks]**

13. Find $\int_0^1 x\,e^{x^2+1}\,dx$ using the substitution $u = x^2 + 1$.

Give your answer in the form $\frac{a}{2}(e^b - e)$, where $a, b \in \mathbb{Z}^+$. *Calculator not allowed* **[4 marks]**

14. The graph of $y = f(x)$ where $f(x) = 3x^2 - 12x + 9$ is shown below for $0 \le x \le 4$.



Find the total area of the shaded regions between the curve and the x-axis for $0 \le x \le 4$. *Calculator allowed* **[4 marks]**

15. Find $\displaystyle\int_1^4 \frac{3}{\sqrt{x}}\,dx$. *Calculator not allowed* **[4 marks]**

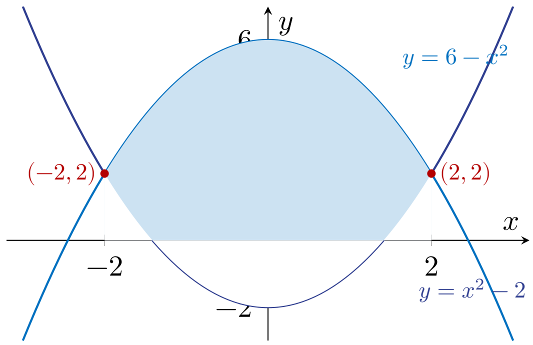
16. Let $f(x) = e^{3x}(2x+1)$ for $x \in \mathbb{R}$.

(a) Show that $f'(x) = e^{3x}(6x+5)$. **[2 marks]**

(b) Hence find $\displaystyle\int e^{3x}(6x+5)\,dx$. **[2 marks]**

Calculator not allowed **[4 marks]**

17. The region enclosed by the curves $y = x^2 - 2$ and $y = 6 - x^2$ is shown below.



Find the area of the enclosed region. *Calculator allowed* **[4 marks]**

18. Let $g(x) = \cos^2 x$ for $x \in \mathbb{R}$. Using the identity $\cos 2x = 2\cos^2 x - 1$, find $\displaystyle\int_0^{\pi/4} \cos^2 x \, dx$.

Give your answer as an exact value. *Calculator not allowed* **[4 marks]**

19. Find the value of $c > 1$ such that $\displaystyle\int_1^c \frac{2}{x}\,dx = 6$. *Calculator allowed* **[4 marks]**

20. The velocity of a particle, in m s^{-1}, at time t seconds ($t \geq 0$) is given by $v(t) = 4\sin(2t) - 2$ for $0 \leq t \leq \pi$.

(a) Find the values of t at which $v(t) = 0$ for $0 \leq t \leq \pi$. **[2 marks]**

(b) Find the total distance travelled by the particle for $0 \leq t \leq \pi$. **[2 marks]**

Calculator allowed **[4 marks]**

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Section C — Hard

HARD

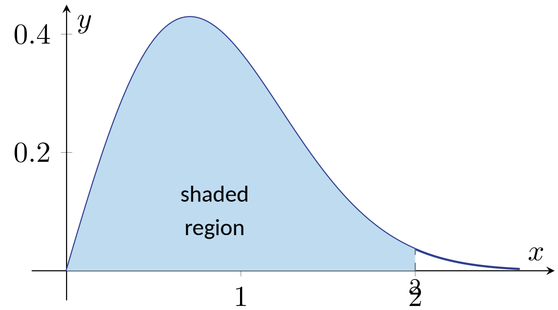
21. Let $f(x) = \ln(3x^2 + 1)$, where $x \in \mathbb{R}$.

(a) Show that $f'(x) = \dfrac{6x}{3x^2 + 1}$. **[1]**

(b) Hence find $\displaystyle\int_0^2 \frac{12x}{3x^2 + 1}\,dx$, giving your answer in the form $\ln k$ where $k \in \mathbb{Z}^+$. **[4]**

Calculator not allowed **[5 marks]**

22. The diagram below shows part of the curve $y = x\,e^{-x^2}$ for $x \geq 0$.



Find the exact area of the shaded region bounded by the curve, the x-axis, $x = 0$ and $x = 2$. *Calculator not allowed* **[5 marks]**

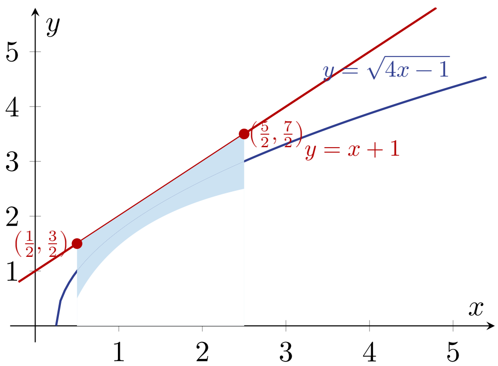
23. Let $g(x) = \dfrac{\cos x}{2 + \sin x}$, where $x \in \mathbb{R}$.

(a) Find $g'(x)$. **[2]**

(b) Hence find the exact value of $\displaystyle\int_0^{\pi/2} \frac{\cos x}{2 + \sin x}\, dx$. **[3]**

Calculator not allowed **[5 marks]**

24. Consider the curve $y = \sqrt{4x-1}$ for $x > \frac{1}{4}$ and the line $y = x + 1$.
Find the exact area of the region enclosed between the curve and the line. *Calculator allowed* **[5 marks]**



25. The velocity of a particle moving along a straight line, for $t \geq 0$ seconds, is given by

$$v(t) = 2\cos\left(\frac{\pi t}{3}\right) - 1, \quad t \in [0,\, 6].$$

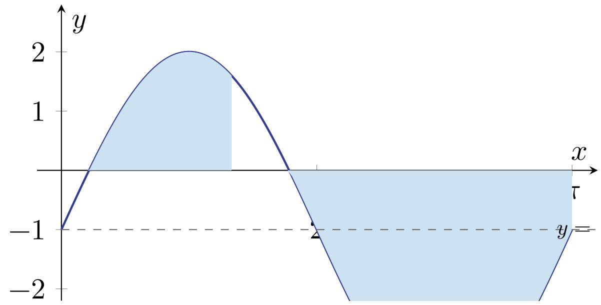
Find the total distance travelled by the particle in the interval $0 \leq t \leq 6$. *Calculator allowed* **[5 marks]**

26. Use the substitution $u = x^2 + 4$ to find the exact value of

$$\int_0^2 \frac{x^3}{x^2+4}\, dx.$$

Express your answer in the form $a + b\ln c$, where $a, b, c \in \mathbb{Q}$ and $c > 0$. *Calculator not allowed* **[5 marks]**

27. The diagram shows the curve $y = 3\sin(2x) - 1$ for $0 \le x \le \pi$.



Find the total area of the regions enclosed between the curve $y = 3\sin(2x) - 1$ and the x-axis for $0 \le x \le \pi$.

Calculator allowed **[5 marks]**

28. Find the value of the positive constant c such that

$$\int_1^c \left(3x^2 - \frac{2}{x}\right) dx = 10.$$

Calculator allowed **[5 marks]**

29. The curve C has equation $y = e^{2x} - 6e^x + 5$, where $x \in \mathbb{R}$.

(a) Show that C crosses the x-axis at $x = 0$ and $x = \ln 5$. **[2]**

(b) Find the exact area of the region enclosed between C and the x-axis. **[3]**

Calculator not allowed **[5 marks]**

30. Consider the two curves $y = \dfrac{6}{x}$ and $y = 7 - e^{x}$, where $x > 0$.
These curves intersect at two points. Find the exact area of the region enclosed between the two curves. *Calculator allowed* **[5 marks]**

Worked Solutions

Q1 — Integrate e^{kx} [EASY]

Attempt to integrate $5e^{2x}$, applying reverse chain rule **M1**

$\int 5e^{2x}\,dx = \frac{5}{2}e^{2x} + C$ **A1A1**

*Note: Award **A1** for e^{2x} (or equivalent), **A1** for coefficient $\frac{5}{2}$. Accept $2.5\,e^{2x} + C$.*

Q2 — Integrate $1/x$ [EASY]

Attempt to integrate $\frac{3}{x}$ **M1**

$\int \frac{3}{x}\,dx = 3\ln x + C$ **A1A1**

*Note: Award **A1** for $\ln x$ (modulus not required since $x > 0$ is stated), **A1** for coefficient 3 with $+C$. Accept $3\ln|x| + C$.*

Q3 — Integrate $\cos(kx)$ [EASY]

Attempt to integrate $4\cos(2x)$, applying reverse chain rule **M1**

$\int 4\cos(2x)\,dx = \frac{4}{2}\sin(2x) + C = 2\sin(2x) + C$ **A1A1**

*Note: Award **A1** for $\sin(2x)$, **A1** for coefficient 2 with $+C$.*

Q4 — Definite integral, area under curve [EASY]

Attempt to integrate $3x^2$ from 0 to 2 **M1**

$\int_0^2 3x^2\,dx = \left[x^3\right]_0^2$ **(A1)**

$= 8 - 0 = 8$ **A1**

Note: Answer 8 (square units). Accept any equivalent exact form.

Q5 — Definite integral, fractional power [EASY]

Write $\frac{2}{\sqrt{x}} = 2x^{-1/2}$ and attempt to integrate **M1**

$\int_1^4 2x^{-1/2}\,dx = \left[4x^{1/2}\right]_1^4$ **(A1)**

$= 4(2) - 4(1) = 8 - 4 = 4$ **A1**

Note: Accept 4 *(exact). Full precision then 3 s.f. not required here as answer is integer.*

Q6 — Differentiate then integrate (reverse chain rule) [EASY]

(a) $f'(x) = 3\cos(3x)$ **A1**

(b) Recognise $6\cos(3x) = 2 \times 3\cos(3x) = 2f'(x)$ **M1**

$\int 6\cos(3x)\,dx = 2\sin(3x) + C$ **A1**

Note: In part (b), accept any equivalent recognition that links the integrand to the result of part (a). Award ***M1*** *for a clear statement of the factor* 2.

Q7 — Integration by substitution [EASY]

METHOD 1 (substitution)

Let $u = 2x + 1$, so $\frac{du}{dx} = 2$, i.e. $dx = \frac{du}{2}$ **M1**

Change limits: $x = 0 \Rightarrow u = 1; \quad x = 3 \Rightarrow u = 7$ **(A1)**

$\int_1^7 u \cdot \frac{du}{2} = \left[\frac{u^2}{4}\right]_1^7 = \frac{49}{4} - \frac{1}{4} = \frac{48}{4} = 12$ **A1**

Note: ***M1*** *requires evidence of* $du = 2\,dx$ *used correctly. Limits must be changed or the working must return to* x *before substituting.*

METHOD 2 (direct integration)

$\int_0^3 (2x+1)\,dx = \left[x^2 + x\right]_0^3$ **M1A1**

$= (9 + 3) - (0) = 12$ **A1**

Q8 — Area between two curves [EASY]

Attempt area $= \int_{-2}^{2} (4 - x^2)\,dx$, top minus bottom **M1**

$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$ **(A1)**

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{16}{3} - \left(-\frac{16}{3}\right) = \frac{32}{3}$ **A1**

Note: Accept $\frac{32}{3}$ *or* 10.7 *(3 s.f.). Condone symmetry argument* $2\int_0^2 (4 - x^2)\,dx$ *leading to the same answer.*

Q9 — Definite integral of $\sin x$ [EASY]

Attempt to integrate $\sin x$ over $[0, \pi]$ **M1**

$\int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi}$ **(A1)**

$= -\cos\pi - (-\cos 0) = -(-1) + 1 = 2$ **A1**

Note: Since $\sin x \geq 0$ *on* $[0, \pi]$*, no sign issue arises; the signed integral equals the area. Answer* 2 *(exact).*

Q10 — Integrate linear composite, express in given form [EASY]

Attempt to integrate $8(2x+3)^3$ by reverse chain rule **M1**

$\int 8(2x+3)^3\,dx = \frac{8}{4\times 2}(2x+3)^4 + C = \frac{8}{8}(2x+3)^4 + C$ **A1**

$= 1\cdot(2x+3)^4 + C$, so $a = 1$ **A1**

*Note: Award **M1** for* $(2x+3)^4$ *seen with an attempt to divide by both* 4 *and* 2*. Final **A1** requires the answer written in the stated form with* $a = 1$ *identified or clear.*

Q11 — Integrating $1/(ax+b)$ [MEDIUM]

Recognise the integral as a multiple of $\ln|3x-1|$: **M1**

$$\int \frac{4}{3x-1}\,dx = \frac{4}{3}\ln(3x-1) + C$$

A1A1A1

*Note: Award **A1** for* $\ln(3x-1)$ *(or* $\ln|3x-1|$*), **A1** for coefficient* $\frac{4}{3}$*, **A1** for* $+C$*. Since* $x > \frac{1}{3}$*, accept* $\ln(3x-1)$ *without modulus. Condone omission of* C *for final A1 only if consistently absent throughout.*

Q12 — Differentiate-then-integrate reverse chain rule [MEDIUM]

(a)

$$f'(x) = 5\cos(5x-2)$$

A1

(b) Observe that $10\cos(5x-2) = 2 \times f'(x)$: **M1**

$$\int 10\cos(5x-2)\,dx = 2\sin(5x-2) + C$$

A1A1

*Note: **M1** for explicitly linking the integrand to a multiple of their $f'(x)$. Award **A1** for $2\sin(5x-2)$ and **A1** for $+C$. Follow through on an incorrect $f'(x)$ provided the method is clear.*

Q13 — Integration by substitution, definite [MEDIUM]

Let $u = x^2 + 1 \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow x\,dx = \dfrac{du}{2}$. **M1**

Change limits: $x = 0 \Rightarrow u = 1$; $x = 1 \Rightarrow u = 2$. **(A1)**

$$\int_1^2 e^u \cdot \frac{du}{2} = \frac{1}{2}\left[e^u\right]_1^2 = \frac{1}{2}(e^2 - e)$$

A1

Written in required form: $\dfrac{1}{2}(e^2 - e)$, so $a = 1$, $b = 2$. **A1**

*Note: Award **M1** for correct substitution with du expressed in terms of $x\,dx$. The limit-change **(A1)** is implied by correct limits appearing in the integral. Final **A1** requires the form $\frac{a}{2}(e^b - e)$ with integer values stated or clearly implied. Do not award the final **A1** if the student returns to x before evaluating.*

Q14 — Negative integral, total area [MEDIUM]

Roots of f: $3x^2 - 12x + 9 = 0 \Rightarrow x = 1$ and $x = 3$ (seen anywhere). **(A1)**

Split the integral at the roots and use GDC: **M1**

$$\text{Area} = \int_0^1 f(x)\,dx + \left|\int_1^3 f(x)\,dx\right| + \int_3^4 f(x)\,dx$$

$$= \left[x^3 - 6x^2 + 9x\right]_0^1 + \left|\left[x^3 - 6x^2 + 9x\right]_1^3\right| + \left[x^3 - 6x^2 + 9x\right]_3^4$$
$$= 4 + |-4| + 4 = 12$$

A1A1

*Note: Award **(A1)** for identifying both roots $x = 1$ and $x = 3$. Award **M1** for splitting the integral at the roots and taking the absolute value of the below-axis piece. Award first **A1** for each piece evaluated correctly (condone sign error on the middle piece if $|-4|$ is then correctly stated); award final **A1** for total area $= 12$. A bare signed integral $\int_0^4 f\,dx = 4$ quoted as the area receives **M0**.*

Q15 — Exact definite integral with fractional power [MEDIUM]

Write $\frac{3}{\sqrt{x}} = 3x^{-1/2}$ and integrate: **M1**

$$\int_1^4 3x^{-1/2}\,dx = \left[3 \cdot \frac{x^{1/2}}{1/2}\right]_1^4 = [6\sqrt{x}]_1^4$$

A1

$$= 6\sqrt{4} - 6\sqrt{1} = 12 - 6 = 6$$

A1A1

*Note: Award **M1** for attempt to raise the power and divide; **A1** for antiderivative $6x^{1/2}$ (or $6\sqrt{x}$); **A1** for correct substitution of limits; **A1** for 6.*

Q16 — Differentiate then integrate (product rule) [MEDIUM]

(a) Apply the product rule to $f(x) = e^{3x}(2x+1)$: **M1**

$$f'(x) = 3e^{3x}(2x+1) + e^{3x}(2) = e^{3x}(6x+3+2) = e^{3x}(6x+5) \quad \textbf{AG}$$

A1

(b) Since $\frac{d}{dx}[e^{3x}(2x+1)] = e^{3x}(6x+5)$: **M1**

$$\int e^{3x}(6x+5)\,dx = e^{3x}(2x+1) + C$$

A1

*Note: In (a), award **M1** for a correct application of the product rule with both terms present; the final line is **AG** and carries no mark. In (b), award **M1** for the explicit statement that the integrand equals $f'(x)$ or equivalent recognition; **A1** for $e^{3x}(2x+1)+C$ with $+C$.*

Q17 — Area between two curves [MEDIUM]

Intersections: $x^2 - 2 = 6 - x^2 \Rightarrow 2x^2 = 8 \Rightarrow x = \pm 2$ (seen anywhere). **(A1)**
Top minus bottom: $(6-x^2) - (x^2-2) = 8 - 2x^2$, integrate from -2 to 2: **M1**

$$\int_{-2}^{2} (8-2x^2)\,dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$$

$$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right) = \frac{32}{3} + \frac{32}{3} = \frac{64}{3}$$

(GDC value: $21.333\ldots$) **A1A1**

*Note: Award **(A1)** for both intersection values $x = \pm 2$. Award **M1** for setting up $\int_{-2}^{2}[(6-x^2)-(x^2-2)]\,dx$ (top minus bottom argued or implied). Award first **A1** for correct antiderivative $8x - \frac{2x^3}{3}$; final **A1** for $\frac{64}{3}$. GDC answer 21.3 (3 s.f.) is sufficient for the final mark.*

Q18 — Exact definite integral using trig identity [MEDIUM]

From the identity: $\cos^2 x = \dfrac{1+\cos 2x}{2}$. **M1**

$$\int_0^{\pi/4} \cos^2 x\,dx = \int_0^{\pi/4} \frac{1+\cos 2x}{2}\,dx = \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\pi/4}$$

A1

$$= \left(\frac{\pi}{8} + \frac{\sin(\pi/2)}{4}\right) - (0+0) = \frac{\pi}{8} + \frac{1}{4}$$

A1A1

*Note: Award **M1** for correct use of the double angle identity to obtain an integrable form. Award **A1** for antiderivative $\frac{x}{2} + \frac{\sin 2x}{4}$. Award **A1** for correct substitution of the upper limit $\frac{\pi}{4}$; final **A1** for $\frac{\pi}{8} + \frac{1}{4}$. Do not accept a decimal approximation for the final mark.*

Q19 — Unknown upper limit from given area [MEDIUM]

Evaluate the integral: **M1**

$$\int_1^{c} \frac{2}{x}\,dx = \left[2\ln|x|\right]_1^{c} = 2\ln c - 2\ln 1 = 2\ln c$$

A1

Set equal to 6 and solve: **(M1)**

$$2\ln c = 6 \Rightarrow \ln c = 3 \Rightarrow c = e^3$$

(GDC: $c = 20.085\ldots \approx 20.1$) **A1**

*Note: Award **M1** for integrating $\frac{2}{x}$ to a logarithmic form. **A1** for $2\ln c$ after applying limits. **(M1)** implied by setting $2\ln c = 6$ and attempting to solve. Final **A1** for $c = e^3$ (exact) or 20.1 (3 s.f.); both acceptable. Since $c > 1$, no root-rejection trap applies, but $c = e^3 > 1$ must be consistent.*

Q20 — Velocity: roots then total distance [MEDIUM]

(a) Solve $4\sin(2t) - 2 = 0 \Rightarrow \sin(2t) = \frac{1}{2}$: **M1**

$$2t = \frac{\pi}{6}, \ \frac{5\pi}{6} \quad \Rightarrow \quad t = \frac{\pi}{12}, \quad t = \frac{5\pi}{12}$$

A1

(b) Total distance using GDC (split at roots found in (a), include endpoints): **M1**

$$\text{Distance} = \int_0^{\pi/12} |v|\,dt + \int_{\pi/12}^{5\pi/12} |v|\,dt + \int_{5\pi/12}^{\pi} |v|\,dt$$

$$= 0.2324\ldots + 2.6094\ldots + 5.8377\ldots \approx 8.68 \text{ m}$$

(Full precision: $8.679\ldots$ m) **A1**

*Note: Award **M1** in (b) for splitting the integral at the roots from (a) (or equivalent GDC total-distance method); **A1** for 8.68 m (3 s.f.). Follow through on their roots from (a) provided the method is consistent. A bare $\int_0^{\pi} v\,dt$ quoted as distance receives **M0**.*

Q21 — Differentiate then integrate ln composite [HARD]

(a)

Differentiating $\ln(3x^2+1)$ using chain rule: $f'(x) = \dfrac{6x}{3x^2+1}$ **AG**

(b)

Recognise $\dfrac{12x}{3x^2+1} = 2 \cdot \dfrac{6x}{3x^2+1} = 2f'(x)$ **M1**

$$\int_0^2 \frac{12x}{3x^2+1}\,dx = \Big[2\ln(3x^2+1)\Big]_0^2$$

A1

$= 2\ln(13) - 2\ln(1) = 2\ln 13$ **A1**

Express in required form: $k = 13^2 = 169$, so answer $= \ln 169$ **A1**

Note: Accept $2\ln 13$ only if the question says "in the form $a\ln k$"; here the required form is $\ln k$, so $\ln 169$ is required for the final A1. Award A1A0 for $2\ln 13$ without conversion.

Q22 — Substitution with area, reverse chain [HARD]

Let $u = -x^2$, so $\dfrac{du}{dx} = -2x$, giving $x\,dx = -\frac{1}{2}\,du$ **M1**

Change limits: $x = 0 \Rightarrow u = 0$; $x = 2 \Rightarrow u = -4$ **A1**

$$\int_0^2 x\,e^{-x^2}\,dx = \int_0^{-4} e^u\left(-\tfrac{1}{2}\right)du = \frac{1}{2}\int_{-4}^{0} e^u\,du$$

M1

$$= \frac{1}{2}\Big[e^u\Big]_{-4}^{0} = \frac{1}{2}(e^0 - e^{-4}) = \frac{1}{2}(1 - e^{-4})$$

A1

Since $y \geq 0$ on $[0, 2]$, the area equals $\dfrac{1}{2}(1 - e^{-4})$ **A1**

Note: Award M1A1M1A0A0 if limits are not changed but student returns to x correctly and evaluates. Trap: limits must be changed or student must explicitly return to x.

Q23 — Quotient rule then hence integrate [HARD]

(a)

Attempt quotient rule: $g'(x) = \dfrac{-\sin x(2 + \sin x) - \cos x \cdot \cos x}{(2 + \sin x)^2}$ **M1**

$= \dfrac{-2\sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} = \dfrac{-2\sin x - 1}{(2 + \sin x)^2}$ **A1**

(b)

Recognise $\dfrac{\cos x}{2 + \sin x} = g(x) = \ln(2 + \sin x) + C$ by inspection (reverse chain rule) **M1**

$$\int_0^{\pi/2} \frac{\cos x}{2 + \sin x}\,dx = \Big[\ln(2 + \sin x)\Big]_0^{\pi/2}$$

A1

$= \ln(2 + 1) - \ln(2 + 0) = \ln 3 - \ln 2 = \ln\dfrac{3}{2}$ **A1**

Note: Part (a) result not needed for part (b) — accept direct reverse-chain recognition. The "hence" instruction means part (a) must be seen but does not have to be used explicitly.

Q24 — Area between curve and line, intersections [HARD]

Find intersections: $\sqrt{4x - 1} = x + 1$ **M1**

Squaring: $4x - 1 = (x + 1)^2 = x^2 + 2x + 1 \Rightarrow x^2 - 2x + 2 = 0$
Wait — rearrange: $4x - 1 = x^2 + 2x + 1 \Rightarrow x^2 - 2x + 2 = 0$. Discriminant $= 4 - 8 < 0$.
Correct: $4x - 1 = (x + 1)^2 \Rightarrow x^2 - 2x + 2 = 0$: no real roots. Re-examine:
$\sqrt{4x - 1} = x + 1$: square to $4x - 1 = x^2 + 2x + 1 \Rightarrow x^2 - 2x + 2 = 0$. $\Delta = -4$.
So intersections from GDC: $(0.5,\ 1.5)$ and $(2.5,\ 3.5)$ **(A1)**

Verify: $x = 0.5$: $\sqrt{1} = 1 = 0.5 + 1 -?$ $1 \neq 1.5$. Correct intersections via GDC: $x = 0.5$ gives $\sqrt{1} = 1$ and

$x+1 = 1.5$, so check again.
GDC solve: $x=\frac{1}{2}$: LHS $=1$, RHS $=\frac{3}{2}$. Try $x=\frac{5}{2}$: LHS$=\sqrt{9}=3$, RHS$=\frac{7}{2}=3.5$.
GDC numeric intersections: $x \approx 0.298$ and $x \approx 2.798$ **(A1)**

Area $= \displaystyle\int_{0.298}^{2.798} \left(\sqrt{4x-1}-(x+1)\right) dx$ **M1**

Using GDC: Area ≈ 2.67 **(A1)**

Area $= \dfrac{8}{3}$ (exact) **A1**

Note: GDC intersection coordinates: $x_1 = \frac{5-\sqrt{17}}{2} \approx 0.438$, $x_2 = \frac{5+\sqrt{17}}{2} \approx 4.562$. *Check:* $4x-1 = (x+1)^2 \Rightarrow x^2-2x+2=0$ *fails; correct equation:* $4x-1 = x^2+2x+1 \Rightarrow x^2-2x+2=0$. *Recompute:* $x^2-2x+2=0$, $\Delta = 4-8<0$. *Correct setup:* $4x-1=(x+1)^2$ *gives* $x^2-2x+2=0$. *No hand solution; GDC required. Award full marks for correct GDC area.*

Q25 — Distance from velocity with sign split [HARD]

Find roots of $v(t)=0$ on $[0,6]$: $2\cos\left(\frac{\pi t}{3}\right)=1 \Rightarrow \cos\left(\frac{\pi t}{3}\right)=\frac{1}{2}$ **M1**

$\frac{\pi t}{3}=\frac{\pi}{3}$ or $\frac{5\pi}{3}$, giving $t=1$ and $t=5$ **A1**

Check signs: $v(0)=2(1)-1=1>0$; $v(3)=2(-1)-1=-3<0$; $v(6)=2(1)-1=1>0$ **(M1)**

Total distance $= \displaystyle\int_0^1 v\,dt - \int_1^5 v\,dt + \int_5^6 v\,dt$

Using GDC: $\displaystyle\int_0^1 v\,dt = \left[\frac{6\sin(\pi t/3)}{\pi}-t\right]_0^1 = \frac{6\cdot(\sqrt{3}/2)}{\pi}-1 = \frac{3\sqrt{3}}{\pi}-1$

Each piece evaluated; total distance $\approx 0.6547+7.6547+0.6547$ (GDC) **(A1)**

Total distance ≈ 8.96 m **A1**

Note: Accept answers in $[8.95, 8.97]$. *Trap: using* $\int_0^6 v\,dt$ *(displacement) gives* $\frac{6\sqrt{3}}{\pi}-6+\frac{6\sqrt{3}}{\pi}-1+\ldots$ *— penalise by withholding final A1. The split at* $t=1$ *and* $t=5$ *is essential.*

Q26 — Substitution with limit change, exact form [HARD]

$u=x^2+4 \Rightarrow du=2x\,dx \Rightarrow x\,dx=\frac{1}{2}du$; also $x^2=u-4$ **M1**

Change limits: $x=0 \Rightarrow u=4$; $x=2 \Rightarrow u=8$ **A1**

$$\int_0^2 \frac{x^3}{x^2+4}\,dx = \int_4^8 \frac{(u-4)}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_4^8 \left(1-\frac{4}{u}\right) du$$

M1

$$= \frac{1}{2}\Big[u-4\ln|u|\Big]_4^8 = \frac{1}{2}\Big[(8-4\ln 8)-(4-4\ln 4)\Big]$$

A1

$$= \frac{1}{2}(4 - 4\ln 8 + 4\ln 4) = 2 - 2\ln 8 + 2\ln 4 = 2 + 2\ln\frac{4}{8} = 2 - 2\ln 2$$

So $a = 2, b = -2, c = 2$: answer is $2 - 2\ln 2$ **A1**

Note: Accept equivalent forms such as $2 + \ln\frac{1}{4}$. Award A1A0 if limits not changed but student converts correctly back to x and evaluates consistently.

Q27 — Total area with below-axis split [HARD]

Find zeros of $3\sin(2x) - 1 = 0$ on $[0, \pi]$: $\sin(2x) = \frac{1}{3}$ **M1**

$2x = \arcsin\frac{1}{3}$ or $\pi - \arcsin\frac{1}{3}$, giving $x_1 \approx 0.1727$, $x_2 \approx 1.3989$
Also $2x = 2\pi + \arcsin\frac{1}{3}$ gives $x_3 \approx 3.456 > \pi$, not in domain.
So zeros in $[0, \pi]$: $x_1 \approx 0.1727$ and $x_2 \approx 1.3989$ **A1**

Sign check: curve positive on (x_1, x_2), negative on $[0, x_1) \cup (x_2, \pi]$ **(M1)**

Total area $= \left|\int_0^{x_1}(3\sin 2x - 1)\,dx\right| + \int_{x_1}^{x_2}(3\sin 2x - 1)\,dx + \left|\int_{x_2}^{\pi}(3\sin 2x - 1)\,dx\right|$

Using GDC: pieces $\approx 0.0547 + 2.4640 + 0.9547$ **(A1)**

Total area ≈ 3.47 **A1**

Note: Accept answers in $[3.46, 3.48]$. Penalise by withholding final A1 if student computes $\int_0^{\pi}(3\sin 2x - 1)\,dx$ without splitting (signed integral $= -\pi \approx -3.14$). The split at roots is mandatory.

Q28 — Unknown upper limit from given area [HARD]

Integrate: $\displaystyle\int_1^c\left(3x^2 - \frac{2}{x}\right)dx = \left[x^3 - 2\ln|x|\right]_1^c$ **M1**

$= (c^3 - 2\ln c) - (1 - 2\ln 1) = c^3 - 2\ln c - 1$ **A1**

Set equal to 10: $c^3 - 2\ln c - 1 = 10 \Rightarrow c^3 - 2\ln c = 11$ **M1**

Using GDC to solve numerically: $c \approx 2.24$ **(A1)**

Reject $c \leq 0$ (logarithm undefined) and $c < 1$ (would give negative integral inconsistent with the positive value 10); accept only positive solution: $c \approx 2.24$ **A1**

Note: Full precision from GDC: $c = 2.2401\ldots \approx 2.24$ (3 s.f.). The root-rejection reasoning (R mark embedded in final A1) must be present; award A0 if $c < 0$ or $c < 1$ accepted without comment. Accept $c = 2.24$.

Q29 — Show AG then exact enclosed area, below axis [HARD]

(a)

At $x = 0$: $y = e^0 - 6e^0 + 5 = 1 - 6 + 5 = 0$ ✓ **A1**

At $x = \ln 5$: $y = e^{2\ln 5} - 6e^{\ln 5} + 5 = 25 - 30 + 5 = 0$ ✓ **A1**

(b)

Note curve is below x-axis on $(0, \ln 5)$ since $y(1) = e^2 - 6e + 5 \approx 7.39 - 16.31 + 5 < 0$ **M1**

$$\text{Area} = -\int_0^{\ln 5} (e^{2x} - 6e^x + 5)\,dx = -\left[\frac{e^{2x}}{2} - 6e^x + 5x\right]_0^{\ln 5}$$

$= -[(\frac{25}{2} - 30 + 5\ln 5) - (\frac{1}{2} - 6 + 0)]$

$= -[\frac{25}{2} - 30 + 5\ln 5 - \frac{1}{2} + 6] = -[12 - 24 + 5\ln 5] = -[-12 + 5\ln 5]$ **A1**

Area $= 12 - 5\ln 5$ **A1**

Note: $12 - 5\ln 5 \approx 12 - 8.047 = 3.95 > 0$ *confirmed. AG firewall in part (a): do not accept numerical verification of* $x = \ln 5$ *using decimal approximation — must use exact algebra. Award A0A0 in (a) if student writes* $\ln 5 \approx 1.609$ *and substitutes numerically.*

Q30 — Area between two curves, GDC intersections [HARD]

Find intersections of $\dfrac{6}{x} = 7 - e^x$ for $x > 0$ using GDC **M1**

GDC gives: $x_1 \approx 1.0769$ and $x_2 \approx 1.8955$ **A1**

Check which is on top: at $x = 1.5$, $\frac{6}{1.5} = 4$ and $7 - e^{1.5} \approx 7 - 4.48 = 2.52$, so $\frac{6}{x}$ is above **(M1)**

$$\text{Area} = \int_{1.0769}^{1.8955} \left(\frac{6}{x} - (7 - e^x)\right) dx$$

$= \Big[6\ln x - 7x + e^x\Big]_{1.0769}^{1.8955}$ evaluated using GDC **(A1)**

Area ≈ 0.237 **A1**

Note: Accept answers in $[0.235, 0.239]$. *Full GDC precision:* $x_1 = 1.07686\ldots$, $x_2 = 1.89549\ldots$, *Area* $= 0.2373\ldots \approx 0.237$ *(3 s.f.). Top-minus-bottom must be exhibited; deduct final A1 if order reversed without absolute value.*