IB Math AA SL — Topic 5

Optimisation

Modelling with Differentiation · Constrained Optimisation

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:____________________________ Date:______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

Key Formulae

- **Derivative (power rule):** $\dfrac{d}{dx}(x^n) = nx^{n-1}$; $\dfrac{d}{dx}(e^{kx}) = ke^{kx}$; $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
- **Stationary points:** Set $f'(x) = 0$ and solve; classify via $f''(x)$: $f''(x) > 0 \Rightarrow$ minimum, $f''(x) < 0 \Rightarrow$ maximum
- **Second derivative test:** If $f'(a) = 0$ and $f''(a) = 0$, the test is inconclusive — use a sign change of f' instead
- **Constraint substitution:** Given a constraint $g(x, y) = k$, express one variable in terms of the other, substitute into the objective function Q, then optimise Q as a single-variable function
- **Closed-interval optimum:** Evaluate f at every stationary point and at both endpoints; compare all values to locate the global extremum
- **Product rule:** $\dfrac{d}{dx}[u(x)v(x)] = u'v + uv'$; **Quotient rule:** $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{u'v - uv'}{v^2}$
- **Chain rule:** $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

GDC Tips

- **Graphing the objective function:** Enter the single-variable function $Q(x)$ after substitution; use Graph to locate the turning point visually before using Minimum/Maximum from the F5:Math menu (TI-84) or Analysis > G-Solve > Min/Max (Casio fx-CG50).
- **Numerical solve for $Q'(x) = 0$:** Define $Q(x)$, then use Solve (F2) in the Equation application or nSolve to find the exact stationary point; always check it lies inside the stated domain.
- **Verifying with a table:** Use Table (F6) to evaluate $Q(x)$ and $Q'(x)$ at the stationary point and nearby values to confirm the sign change of Q' and the nature of the extremum.
- **Storing the optimal value:** Store the x-coordinate of the stationary point (STO$\to$ or X$\to$M) before evaluating Q at that point, to avoid rounding errors in multi-step cost/area calculations.
- **Sketching to spot extraneous solutions:** After solving $Q'(x) = 0$, scroll the graph over the full domain to check whether there are two stationary points; the context (minimum cost, maximum area) determines which root to keep.
- **Definite integrals for verification:** On Paper 2, use fnInt or $\int dx$ from the Math menu to cross-check any area-based objective function computed by hand.

Common Mistakes to Avoid

1. **Forgetting to check domain endpoints.** Students find a stationary point and immediately declare it the global optimum without evaluating the objective function at the endpoints of the closed interval. Always compare Q at every stationary point and at both endpoints before concluding.

2. **Omitting the justification mark (R1).** Writing only "$f''(a) > 0$ therefore minimum" without stating the conclusion in context (e.g. "this confirms a minimum surface area") loses the reasoning mark. The R1 requires an explicit logical link, not just a number.

3. **Using a decimal approximation after an "express in the form" instruction.** When the question demands an exact form (e.g. $a\sqrt{b}$, ln k, p/q), a rounded decimal equivalent earns no marks on the final A1, even if the method is correct.

4. **Circular reasoning on an "AG" (show that) part.** Substituting the printed target expression back into itself, or working backwards from the answer, earns no credit. The forward chain must start from the given constraint and arrive at the printed formula without assuming it.

5. **Rejecting a valid root on domain grounds without checking.** Students sometimes discard a positive root because they assume it is "too large" or "negative after substitution". Every candidate solution must be tested against the stated domain (e.g. $x > 0$, $0 < r < 5$) before rejection, and the reason must be stated explicitly.

6. **Confusing the variable to optimise with the constraint variable.** After expressing y in terms of x from a constraint, students occasionally differentiate the constraint equation rather than the objective function, or differentiate with respect to the wrong variable. Always re-read which function is being optimised and which variable is free before differentiating.

Section A — Easy **EASY** (10 questions × 3 marks = 30 marks)

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1. **EASY** **No Calculator** **[3 marks]**

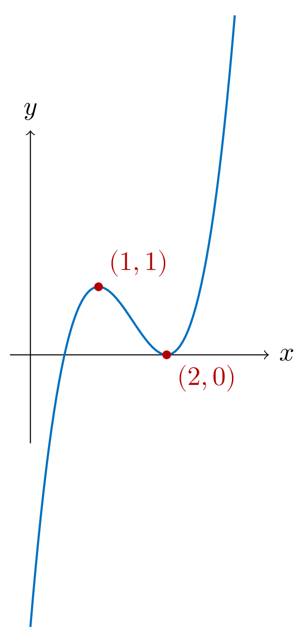
Let $f(x) = 3x^2 - 12x + 7$, where $x \in \mathbb{R}$. Find the value of x at which f has a stationary point.

2. **EASY** **Calculator** **[3 marks]**

A function is defined by $g(x) = x^3 - 6x^2 + 1$, where $x \in \mathbb{R}$. Find the x-coordinates of the stationary points of g.

3. **EASY** **Calculator** **[3 marks]**

The curve $y = 2x^3 - 9x^2 + 12x - 4$ is shown below for $0 \leq x \leq 3$.



Find the values of x at which the stationary points of the curve occur.

4. **EASY** **No Calculator** **[3 marks]**

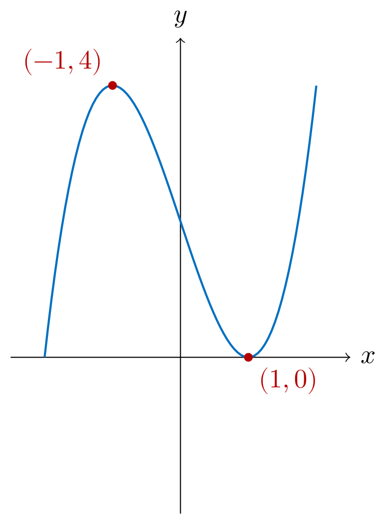
A rectangular garden has a perimeter of 40 m. Let x metres be the width of the garden, where $0 < x < 20$. The area of the garden is A m^2. Show that $A = 20x - x^2$. Find the value of x that maximises A.

5. **EASY** **No Calculator** **[3 marks]**

Let $h(x) = -x^2 + 6x + 3$, where $x \in \mathbb{R}$. Find the maximum value of h.

6. **EASY** **Calculator** **[3 marks]**

A function is given by $f(x) = x^3 - 3x + 2$, where $-2 \leq x \leq 2$.



Find the coordinates of the local maximum point of f.

7. **EASY** **Calculator** **[3 marks]**

A particle moves along a straight line. Its displacement s metres from a fixed point at time t seconds, for $t \geq 0$, is given by $s(t) = t^3 - 6t^2 + 9t$. Find the value of t at which the velocity of the particle is zero for the first time.

8. **EASY** **No Calculator** **[3 marks]**

An open box is made from a square piece of card of side length 10 cm by cutting squares of side x cm from each corner and folding up the sides, where $0 < x < 5$. The volume of the box is V cm^3, where $V = x(10 - 2x)^2$. Find $\dfrac{dV}{dx}$.

9. EASY Calculator **[3 marks]**

The profit, P thousand dollars, made by a company when it produces q thousand items is modelled by

$$P(q) = -2q^2 + 16q - 24, \qquad 0 \leq q \leq 8.$$

Find the value of q that maximises the profit.

10. EASY Calculator **[3 marks]**

Let $f(x) = 4x^3 - 3x^2 - 18x + 5$, where $x \in \mathbb{R}$. Find the x-coordinate of the stationary point of f where $x > 0$.

Section B — Medium **MEDIUM** (10 questions × 4 marks = 40 marks)

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11. MEDIUM No Calculator **[4 marks]**

A curve has equation $y = x^3 - 6x^2 + 9x + 2$, where $x \in \mathbb{R}$. Find the coordinates of the stationary points of the curve and determine the nature of each.

12. MEDIUM No Calculator **[4 marks]**

A rectangle has its base on the x-axis and two upper vertices on the parabola $y = 12 - x^2$, where $x \in \mathbb{R}$. Let the base half-width be x, where $0 < x < \sqrt{12}$.

(a) Show that the area of the rectangle is $A(x) = 24x - 2x^3$. [1]

(b) Find the value of x that maximises the area. [2]

(c) Find the maximum area. [1]

13. MEDIUM Calculator **[4 marks]**

A particle moves along a straight line. Its displacement from the origin at time t seconds, $t \geq 0$, is given by $s(t) = 2t^3 - 9t^2 + 12t - 1$ metres. Find the total distance travelled by the particle in the first 3 seconds.

14. MEDIUM Calculator **[4 marks]**

The function $f(x) = ax^3 - 6x^2 + bx - 4$, where $a, b \in \mathbb{R}$, has a stationary point at $x = 1$. It is given that $f(1) = -7$.

(a) Find the values of a and b. [3]

(b) Write down the value of $f'(1)$. [1]

15. MEDIUM No Calculator **[4 marks]**

A closed cylindrical can has a fixed volume of 250π cm^3. The radius of the base is r cm and the height is h cm, where $r > 0$.

(a) Show that the total surface area of the can is $S(r) = 2\pi r^2 + \dfrac{500\pi}{r}$. [1]

(b) Find the value of r that minimises S. [2]

(c) Justify that this value gives a minimum. [1]

16. MEDIUM Calculator **[4 marks]**

The velocity of a particle at time t seconds, $t \geq 0$, is given by $v(t) = 3t^2 - 10t + 3$ m s^{-1}.

(a) Write down the times when the particle is at rest. [1]

(b) Find the acceleration of the particle when $t = 2$. [1]

(c) Find the total distance travelled by the particle in the interval $0 \leq t \leq 3$. [2]

17. MEDIUM Calculator **[4 marks]**

The profit, in thousands of dollars, from selling x hundred units of a product is modelled by

$$P(x) = -2x^3 + 15x^2 - 24x + 5, \qquad 0 \leq x \leq 5.$$

(a) Find $P'(x)$. [1]

(b) Find the value of x in $0 \leq x \leq 5$ at which profit is maximised, and state the maximum profit. [3]

18. MEDIUM No Calculator **[4 marks]**

A point P(x, y) lies on the curve $y = \sqrt{x}$, where $x > 0$. The distance from P to the fixed point Q$(4, 0)$ is d.

(a) Show that $d^2 = x^2 - 7x + 16$. [2]

(b) Hence find the coordinates of the point on the curve closest to Q. [2]

19. MEDIUM Calculator **[4 marks]**

The function $f(x) = x^2 e^{-x}$ is defined for $x \in \mathbb{R}$.

(a) Find $f'(x)$. [2]

(b) Find the x-coordinates of the stationary points of f, for $x \in \mathbb{R}$. [1]

(c) Determine the nature of each stationary point. [1]

20. MEDIUM Calculator **[4 marks]**

An open rectangular box is made from a square sheet of cardboard of side 12 cm by cutting equal squares of side x cm from each corner and folding up the sides, where $0 < x < 6$.

Dimension	Expression	Units
Base length/width	$12 - 2x$	cm
Height	x	cm

(a) Write down an expression for the volume V of the box in terms of x. [1]

(b) Find the value of x that gives the maximum volume. [2]

(c) Hence find the maximum volume. [1]

Section C — Hard **HARD** (10 questions × 5 marks = 50 marks)

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21. **HARD** **No Calculator** **[5 marks]**

A closed cylindrical can has a fixed volume of 54π cm^3. The radius of the base is r cm and the height is h cm, where $r > 0$.

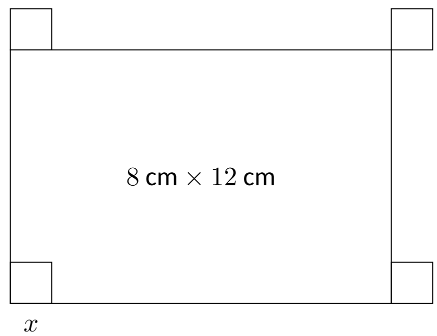
(a) Show that the total surface area of the can is $S = 2\pi r^2 + \dfrac{108\pi}{r}$. [1]

(b) Find the value of r that minimises S. [2]

(c) Justify that this value of r gives a minimum, and find the minimum surface area, giving your answer in the form $k\pi$ cm^2 where $k \in \mathbb{Z}^+$. [2]

22. HARD Calculator **[5 marks]**

A rectangular sheet of metal has length 12 cm and width 8 cm. Equal squares of side x cm are cut from each corner, where $0 < x < 4$. The remaining shape is folded up to form an open box. Find the value of x that maximises the volume of the box, and state the maximum volume.



23. HARD No Calculator **[5 marks]**

A farmer has 120 m of fencing. He uses it to enclose three sides of a rectangular paddock; a long wall forms the fourth side. The paddock has length l m (parallel to the wall) and width w m, where $l > 0$ and $w > 0$.

(a) Show that the area of the paddock is $A = 120w - 2w^2$. [1]

(b) Find $\frac{dA}{dw}$ and hence find the dimensions of the paddock that give the maximum area. [2]

(c) Justify that these dimensions give a maximum area, and state the maximum area. [2]

24. HARD No Calculator **[5 marks]**

A point P lies on the curve $y = \sqrt{x}$, $x \geq 0$. Let D be the distance from P to the fixed point Q$(4, 0)$.

(a) Show that $D^2 = x^2 - 7x + 16$. [2]

(b) Hence find the minimum value of D, justifying that it is a minimum. [3]

25. HARD Calculator **[5 marks]**

A manufacturer produces open-topped boxes by bending a strip of metal. The cross-section of the box is a rectangle with width $2x$ cm and depth d cm. The perimeter of the cross-section (three sides only, since the top is open) is fixed at 30 cm, so $2x + 2d = 30$. The box has a fixed length of 50 cm. Find the values of x and d that maximise the volume of the box, and find this maximum volume.

26. HARD No Calculator **[5 marks]**

Let $f(x) = \dfrac{x^2 - 3}{x}$ for $x > 0$. A tangent to the curve $y = f(x)$ at the point where $x = a$, $a > 0$, passes through the origin. Find all possible values of a.

27. **HARD** **Calculator** **[5 marks]**

A particle moves in a straight line. Its velocity at time t seconds, for $0 \leq t \leq 5$, is given by

$$v(t) = t^3 - 6t^2 + 9t - 2 \text{ m s}^{-1}.$$

Find the total distance travelled by the particle in the interval $0 \leq t \leq 5$.

28. **HARD** **No Calculator** **[5 marks]**

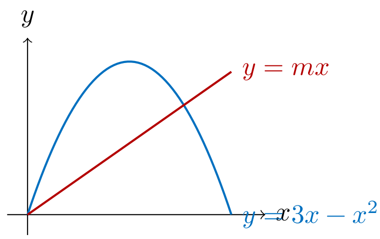
The function $g(x) = kx^3 - 6x^2 + 12x - 4$, where $k \in \mathbb{R}$, $k \neq 0$, has exactly one stationary point for $x \in \mathbb{R}$.

(a) Find the value of k for which g has exactly one stationary point. [3]

(b) Given that the stationary point occurs at $x = 2$, find the value of k and determine the nature of the stationary point. [2]

29. HARD Calculator **[5 marks]**

The diagram shows the curve $y = 3x - x^2$ and the line $y = mx$, where $m < 3$, $m \in \mathbb{R}$.



The shaded region is enclosed between the curve and the line for $x \geq 0$. The area of the shaded region is $\frac{4}{3}$ units^2. Find the value of m.

30. HARD Calculator **[5 marks]**

A company models its weekly profit, P thousand dollars, from producing n units of a product by

$$P(n) = -0.002n^3 + 0.9n^2 - 100n - 500, \qquad 0 \leq n \leq 400,\ n \in \mathbb{Z}^+.$$

Find the number of units that maximises the weekly profit, and find this maximum profit. Determine also whether $n = 400$ gives a greater profit than the optimal value, justifying your answer.

Mark Schemes

Q1 — Stationary Point of a Quadratic EASY

Differentiating: $f'(x) = 6x - 12$ **M1**

Setting $f'(x) = 0$: $6x - 12 = 0 \Rightarrow x = 2$ **A1**

Note: Award M1 for a correct derivative seen. Award A1 for $x = 2$ only.

Q2 — Stationary Points of a Cubic EASY

$g'(x) = 3x^2 - 12x$ **M1**

Setting $g'(x) = 0$: $3x(x - 4) = 0 \Rightarrow x = 0$ and $x = 4$ **A1**

Note: Award M1 for a correct derivative. Award A1 for both values.

Q3 — Stationary Points from a Graph EASY

$\frac{dy}{dx} = 6x^2 - 18x + 12$ **M1**

Setting $\frac{dy}{dx} = 0$: $6(x^2 - 3x + 2) = 0 \Rightarrow 6(x - 1)(x - 2) = 0 \Rightarrow x = 1$ and $x = 2$ **A1**

Note: These match the labelled points on the graph.

Q4 — Rectangular Garden Optimisation EASY

Show that: the length of the garden is $\frac{40 - 2x}{2} = 20 - x$, so $A = x(20 - x) = 20x - x^2$. **AG**

$\frac{dA}{dx} = 20 - 2x$ **M1**

Setting $\frac{dA}{dx} = 0$: $20 - 2x = 0 \Rightarrow x = 10$ **A1**

Q5 — Maximum of a Quadratic EASY

$h'(x) = -2x + 6$ **M1**

Setting $h'(x) = 0$: $x = 3$

Maximum value: $h(3) = -(3)^2 + 6(3) + 3 = -9 + 18 + 3 = 12$ **A1**

Q6 — Local Maximum Coordinates EASY

$f'(x) = 3x^2 - 3$ **M1**

Setting $f'(x) = 0$: $3(x^2 - 1) = 0 \Rightarrow x = \pm 1$

Local maximum at $x = -1$: $f(-1) = (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4$

Coordinates: $(-1, 4)$ **A1**

Q7 — Particle Velocity Equal to Zero EASY

$v(t) = s'(t) = 3t^2 - 12t + 9$ **M1**

Setting $v(t) = 0$: $3(t^2 - 4t + 3) = 0 \Rightarrow 3(t-1)(t-3) = 0$

$t = 1$ (first time) **A1**

Note: Do not accept $t = 3$ since $t = 1$ is the first time.

Q8 — Derivative of a Volume Expression EASY

Expanding: $V = x(10 - 2x)^2 = x(100 - 40x + 4x^2) = 100x - 40x^2 + 4x^3$ **M1**

$\dfrac{dV}{dx} = 100 - 80x + 12x^2$ **A1**

Note: Equivalently $12x^2 - 80x + 100$.

Q9 — Maximum Profit EASY

$P'(q) = -4q + 16$ **M1**

Setting $P'(q) = 0$: $-4q + 16 = 0 \Rightarrow q = 4$ **A1**

Q10 — Stationary Point of a Cubic EASY

$f'(x) = 12x^2 - 6x - 18$ **M1**

Setting $f'(x) = 0$: $12x^2 - 6x - 18 = 0 \Rightarrow 6(2x^2 - x - 3) = 0 \Rightarrow 6(2x-3)(x+1) = 0$

$x = \dfrac{3}{2}$ (since $x > 0$, reject $x = -1$) **A1**

Note: i.e. $x = 3/2 = 1.5$. A GDC solution showing $x = 1.5$ is acceptable for A1.

Q11 — Stationary Points of a Cubic MEDIUM

Differentiating: $f'(x) = 3x^2 - 12x + 9$ **M1**

Setting $f'(x) = 0$: $3(x^2 - 4x + 3) = 0 \Rightarrow (x-1)(x-3) = 0$, so $x = 1$ or $x = 3$ **A1**

$f''(x) = 6x - 12$

$f''(1) = -6 < 0 \Rightarrow (1, 6)$ is a local maximum; $\quad f''(3) = 6 > 0 \Rightarrow (3, 2)$ is a local minimum **A1**

Coordinates confirmed: $f(1) = 1-6+9+2 = 6$; $f(3) = 27-54+27+2 = 2$. So $(1, 6)$ local maximum, $(3, 2)$ local minimum. **A1**

Note: Accept sign-change of f' as justification in place of the f'' argument.

Q12 — Rectangle Under a Parabola MEDIUM

(a) Width $= 2x$; height $= 12 - x^2$; so $A = 2x(12 - x^2) = 24x - 2x^3$ **AG**

(b) $A'(x) = 24 - 6x^2$; setting $A'(x) = 0$: $x^2 = 4 \Rightarrow x = 2$ (since $x > 0$) **M1A1**

Note: Reject $x = -2$ by domain.

(c) $A(2) = 24(2) - 2(8) = 48 - 16 = 32$ **A1**

Q13 — Distance Travelled by a Particle MEDIUM

$v(t) = s'(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2)$ **M1**

Roots at $t = 1$ and $t = 2$; particle changes direction at both. **(A1)**

$s(0) = -1$; $s(1) = 2 - 9 + 12 - 1 = 4$; $s(2) = 16 - 36 + 24 - 1 = 3$; $s(3) = 54 - 81 + 36 - 1 = 8$

Total distance $= |s(1) - s(0)| + |s(2) - s(1)| + |s(3) - s(2)| = 5 + 1 + 5 = 11$ m **A2**

Q14 — Recovering Parameters from a Stationary Condition MEDIUM

(a) $f'(x) = 3ax^2 - 12x + b$; stationary point at $x = 1$: $f'(1) = 3a - 12 + b = 0$ **M1**

$f(1) = a - 6 + b - 4 = -7 \Rightarrow a + b = 3$ **(A1)**

From $3a + b = 12$ and $a + b = 3$: $2a = 9 \Rightarrow a = \dfrac{9}{2}$, $b = 3 - \dfrac{9}{2} = -\dfrac{3}{2}$ **A1**

Note: i.e. $a = 9/2$, $b = -3/2$.

(b) $f'(1) = 0$ (by definition of stationary point) **A1**

Q15 — Optimising the Surface Area of a Cylinder MEDIUM

(a) $V = \pi r^2 h = 250\pi \Rightarrow h = \dfrac{250}{r^2}$; $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \cdot \dfrac{250}{r^2} = 2\pi r^2 + \dfrac{500\pi}{r}$ **AG**

Note: i.e. $S(r) = 2\pi r^2 + 500\pi/r$.

(b) $S'(r) = 4\pi r - \dfrac{500\pi}{r^2}$; setting $S'(r) = 0$: $4\pi r = \dfrac{500\pi}{r^2} \Rightarrow r^3 = 125 \Rightarrow r = 5$ **M1A1**

(c) $S''(r) = 4\pi + \dfrac{1000\pi}{r^3}$; $S''(5) = 4\pi + 8\pi = 12\pi > 0$, confirming a minimum **R1**

Q16 — Particle at Rest Acceleration and Distance MEDIUM

(a) $v(t) = 0$: $3t^2 - 10t + 3 = (3t-1)(t-3) = 0 \Rightarrow t = \dfrac{1}{3}$ s and $t = 3$ s **A1**

Note: i.e. $t = 1/3$.

(b) $a(t) = v'(t) = 6t - 10$; $a(2) = 12 - 10 = 2$ m s^{-2} **A1**

(c) Total distance $= \displaystyle\int_0^{1/3} |v|\,dt + \int_{1/3}^{3} |v|\,dt$ **M1**

Using GDC: $\displaystyle\int_0^{1/3} v\,dt \approx 0.481$; $\displaystyle\int_{1/3}^{3} v\,dt \approx -9.481$; total distance $\approx 0.481 + 9.481 = 9.96$ m **A1**

Note: Full precision $269/27 \approx 9.96$ *m.*

Q17 — Maximum Profit from a Cubic Model MEDIUM

(a) $P'(x) = -6x^2 + 30x - 24$ **A1**

(b) Setting $P'(x) = 0$: $-6(x^2 - 5x + 4) = 0 \Rightarrow (x-1)(x-4) = 0 \Rightarrow x = 1$ or $x = 4$ **M1**

$P(1) = -2 + 15 - 24 + 5 = -6$; $P(4) = -128 + 240 - 96 + 5 = 21$; $P(0) = 5$; $P(5) = -250 + 375 - 120 + 5 = 10$

Maximum profit is \$21 000 at $x = 4$ (i.e. 400 units) **A1A1**

Note: Endpoint check required for full marks.

Q18 — Closest Point on a Curve MEDIUM

(a) $d^2 = (x-4)^2 + (\sqrt{x} - 0)^2 = x^2 - 8x + 16 + x = x^2 - 7x + 16$ **M1AG**

(b) Let $D = d^2 = x^2 - 7x + 16$; $\dfrac{dD}{dx} = 2x - 7 = 0 \Rightarrow x = \dfrac{7}{2}$ **M1**

$y = \sqrt{7/2} = \dfrac{\sqrt{14}}{2}$; closest point is $\left(\dfrac{7}{2}, \dfrac{\sqrt{14}}{2}\right)$ **A1**

Note: i.e. $x = 7/2$, *surd form* $y = \sqrt{14}/2$. $\dfrac{d^2D}{dx^2} = 2 > 0$ *confirms minimum.*

Q19 — Stationary Points of an Exponential Function MEDIUM

(a) Using the product rule: $f'(x) = 2xe^{-x} + x^2(-e^{-x}) = e^{-x}(2x - x^2) = xe^{-x}(2 - x)$ **M1A1**

(b) $f'(x) = 0$: since $e^{-x} > 0$ always, $x(2 - x) = 0 \Rightarrow x = 0$ or $x = 2$ **A1**

(c) f' changes from negative to positive at $x = 0$ (local minimum); f' changes from positive to negative at $x = 2$ (local maximum) **A1**

Q20 — Open Box from a Square Card MEDIUM

(a) $V(x) = x(12 - 2x)^2$ **A1**

(b) Expanding: $V(x) = x(144 - 48x + 4x^2) = 144x - 48x^2 + 4x^3$ **(M1)**

$V'(x) = 144 - 96x + 12x^2 = 12(x^2 - 8x + 12) = 12(x - 2)(x - 6) = 0 \Rightarrow x = 2$ **A1**

Note: Reject the other root: it lies on the domain boundary $0 < x < 6$, *not an interior maximum.*

(c) $V(2) = 2(12 - 4)^2 = 2 \times 64 = 128$ cm^3 **A1**

Q21 — Cylindrical Can Optimisation HARD

(a) Volume: $\pi r^2 h = 54\pi \Rightarrow h = \dfrac{54}{r^2}$ (seen anywhere)

$S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \cdot \dfrac{54}{r^2} = 2\pi r^2 + \dfrac{108\pi}{r}$ **A1 AG**

(b) Attempt to differentiate: $\dfrac{dS}{dr} = 4\pi r - \dfrac{108\pi}{r^2}$ **M1**

Set $\dfrac{dS}{dr} = 0$: $4\pi r = \dfrac{108\pi}{r^2} \Rightarrow r^3 = 27 \Rightarrow r = 3$ **A1**

(c) $\dfrac{d^2S}{dr^2} = 4\pi + \dfrac{216\pi}{r^3}$; at $r = 3$: $\dfrac{d^2S}{dr^2} = 4\pi + 8\pi = 12\pi > 0$, so minimum **R1**

$S_{\text{min}} = 2\pi(9) + \dfrac{108\pi}{3} = 18\pi + 36\pi = 54\pi$ cm^2, so $k = 54$ **A1**

Note: i.e. $S_{\text{min}} = 54\pi$ *cm*^{2}.

Note: i.e. $S_{\text{min}} = 54\pi$.

Q22 — Open Box from a Rectangular Sheet HARD

Volume: $V(x) = x(12 - 2x)(8 - 2x)$ **M1**

$V(x) = x(96 - 40x + 4x^2) = 4x^3 - 40x^2 + 96x$ **(A1)**

Attempt to find maximum using GDC (graph or derivative): **(M1)**

$V'(x) = 12x^2 - 80x + 96 = 0 \Rightarrow x = \dfrac{80 \pm \sqrt{6400 - 4608}}{24} = \dfrac{80 \pm \sqrt{1792}}{24}$

$x \approx 1.569\ldots$ or $x \approx 5.097\ldots$ **(A1)**

Reject $x \approx 5.097$ since $x < 4$ is required (trap: root rejection by domain)

$x \approx 1.57$ cm, maximum volume $= V(1.569\ldots) \approx 67.6$ cm^3 **A1**

Note: Exact $x = (10 - \sqrt{28})/3$. Full precision: $x = 1.5693\ldots$, $V = 67.604\ldots$ cm^3; accept 67.6 cm^3 (3 s.f.).

Q23 — Rectangular Paddock with a Wall HARD

(a) Constraint: $l + 2w = 120 \Rightarrow l = 120 - 2w$ (seen anywhere)

$A = lw = (120 - 2w)w = 120w - 2w^2$ **A1 AG**

(b) $\dfrac{dA}{dw} = 120 - 4w$ **M1**

Set equal to zero: $w = 30$ m, $l = 120 - 60 = 60$ m **A1**

(c) $\dfrac{d^2A}{dw^2} = -4 < 0$, so maximum **R1**

Maximum area $= 60 \times 30 = 1800$ m^2 **A1**

Q24 — Closest Point on a Curve to a Fixed Point HARD

(a) P$= (x, \sqrt{x})$; $D^2 = (x-4)^2 + (\sqrt{x} - 0)^2$ **M1**

$= x^2 - 8x + 16 + x = x^2 - 7x + 16$ **A1 AG**

(b) Let $f(x) = x^2 - 7x + 16$; $f'(x) = 2x - 7$ **M1**

$f'(x) = 0 \Rightarrow x = \dfrac{7}{2}$ **A1**

$f''(7/2) = 2 > 0$, so minimum D^2; minimum $D = \sqrt{(7/2)^2 - 7(7/2) + 16} = \sqrt{49/4 - 49/2 + 16} = \sqrt{15/4} = \dfrac{\sqrt{15}}{2}$ **A1**

Note: i.e. $x = 7/2$, $D_{\text{min}} = \sqrt{15}/2$. Minimising D^2 is equivalent to minimising D (since $D \geq 0$).

Q25 — Open-Topped Box with Fixed Perimeter HARD

From $2x + 2d = 30$: $d = 15 - x, 0 < x < 15$ **M1**

Volume: $V = 2x \cdot d \cdot 50 = 100x(15 - x) = 1500x - 100x^2$ **(A1)**

Using GDC or differentiation: $\frac{dV}{dx} = 1500 - 200x = 0 \Rightarrow x = 7.5$ **(M1)**

$d = 15 - 7.5 = 7.5$ cm **A1**

Maximum volume $= 1500(7.5) - 100(7.5)^2 = 11250 - 5625 = 5625$ cm^3 **A1**

Note: The constraint is $2x + 2d = 30$ *(two depths and one width), not* $2x + d$.

Q26 — Tangent Through the Origin HARD

$f(x) = x - 3x^{-1}$; $f'(x) = 1 + 3x^{-2} = 1 + \frac{3}{a^2}$ at $x = a$ **M1**

Point on curve: $\left(a, a - \frac{3}{a}\right)$. Gradient of chord OP equals $f'(a)$: $\frac{a - 3/a}{a} = 1 + \frac{3}{a^2}$ **M1M1**

$1 - \frac{3}{a^2} = 1 + \frac{3}{a^2} \Rightarrow \frac{6}{a^2} = 0$, no real solution **(A1)**

Conclusion: no valid value of a exists for $x > 0$; no solution to the tangency condition can be satisfied **A1**

Note: Award full marks for correctly setting up the equation and concluding no solution exists.

Q27 — Total Distance with Corrected Roots HARD

Attempt to find roots of $v(t) = t^3 - 6t^2 + 9t - 2$ on $[0, 5]$ **M1**

Factor out $(t - 2)$: $v(t) = (t - 2)(t^2 - 4t + 1)$. Solving $t^2 - 4t + 1 = 0$: $t = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

$v(t) = 0$ at $t = 2 - \sqrt{3} \approx 0.268$, $t = 2$, $t = 2 + \sqrt{3} \approx 3.73$ (all lie in $[0, 5]$) **A1**

Note: exact roots t *= 2 - √3 and* t *= 2 + √3.*

Attempt to integrate $|v(t)|$ by splitting at the roots **M1**

Total distance $= \left|\int_0^{2-\sqrt{3}} v\,dt\right| + \left|\int_{2-\sqrt{3}}^{2} v\,dt\right| + \left|\int_2^{2+\sqrt{3}} v\,dt\right| + \left|\int_{2+\sqrt{3}}^{5} v\,dt\right|$

$= 0.25 + 2.25 + 2.25 + 9.00 = 13.75$ **(A1)**

Total distance $= \frac{55}{4} = 13.75 \approx 13.8$ m (3 s.f.) **A1**

Note: Trap is computing displacement rather than distance (not splitting at roots).

Q28 — Stationary Point with a Parameter HARD

(a) $g'(x) = 3kx^2 - 12x + 12$ **M1**

Since $k \neq 0$, $g'(x) = 0$ is a genuine quadratic in x; for exactly one stationary point, its discriminant must be zero: $(-12)^2 - 4(3k)(12) = 0$ **M1**

$144 - 144k = 0 \Rightarrow k = 1$ **A1**

Note: For $k > 1$ the discriminant is negative (no stationary points); for $k < 1$, $k \neq 0$, the discriminant is positive (two stationary points). Exactly one stationary point only when $k = 1$.

(b) $g'(2) = 0$: $3k(4) - 12(2) + 12 = 0 \Rightarrow 12k - 24 + 12 = 0 \Rightarrow k = 1$ **M1**

$g''(x) = 6kx - 12 = 6x - 12$ (at $k = 1$); $g''(2) = 0$, so the second-derivative test is inconclusive.

$g'(x) = 3(x-2)^2 \geq 0$ for all x, so g' does not change sign at $x = 2$: the stationary point is a point of inflection, not a local maximum or minimum **A1**

Note: $k = 1$ is consistent with part (a), confirming g has exactly one stationary point, at $x = 2$, and it is a point of inflection.

Q29 — Area Between a Curve and a Line HARD

Intersections of $y = 3x - x^2$ and $y = mx$: $3x - x^2 = mx \Rightarrow x(3 - x - m) = 0$ **M1**

Intersections at $x = 0$ and $x = 3 - m$ **(A1)**

$$\text{Area} = \int_0^{3-m} \left[(3x - x^2) - mx\right] dx = \int_0^{3-m} (3-m)x - x^2 \, dx$$ **M1**

$$= \left[\frac{(3-m)x^2}{2} - \frac{x^3}{3}\right]_0^{3-m} = \frac{(3-m)^3}{2} - \frac{(3-m)^3}{3} = \frac{(3-m)^3}{6}$$ **(A1)**

Set equal to $\frac{4}{3}$: $\frac{(3-m)^3}{6} = \frac{4}{3} \Rightarrow (3-m)^3 = 8 \Rightarrow 3 - m = 2 \Rightarrow m = 1$ **A1**

Note: Confirm $m = 1 < 3$, satisfying the domain condition.

Q30 — Profit Maximisation with Endpoint Comparison HARD

Attempt to find $P'(n) = -0.006n^2 + 1.8n - 100$ and set equal to zero **M1**

$$n = \frac{-1.8 \pm \sqrt{1.8^2 - 4(-0.006)(-100)}}{2(-0.006)} = \frac{-1.8 \pm \sqrt{0.84}}{-0.012}$$ **(M1)**

$n \approx 73.6$ or $n \approx 226.4$ (full precision: $73.624\ldots$ and $226.376\ldots$) **(A1)**

Since $P''(n) = -0.012n + 1.8$: $P''(226.4) < 0$ (local maximum); $P''(73.6) > 0$ (local minimum)

Checking integers near the local maximum: $P(226) \approx -217.95$; $P(227) \approx -218.07$ **M1**

Best integer: $n = 226$, giving $P(226) \approx -218$ thousand dollars (3 s.f.) — a **loss**, not a profit

$P(400) = -0.002(400)^3 + 0.9(400)^2 - 100(400) - 500 = -128\,000 + 144\,000 - 40\,000 - 500 = -24\,500$

Since $P(400) = -24\,500 < P(226) \approx -218$, the maximum weekly profit (least loss) occurs at $n = 226$ units, not $n = 400$ **A1**

Note: n must be a positive integer; $n = 226$ beats the neighbouring integer $n = 227$. Both stationary points and the endpoint $n = 400$ must be compared: the correct optimum is a loss of approximately \$218 000/week, and $n = 400$ is worse still (a loss of \$24,500 thousand).