IB Math AA SL — Topic 5

Optimisation

Modelling with Differentiation · Constrained Optimisation

Difficulty	**EASY**	**MEDIUM**	**HARD**
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. Each question states whether a calculator is permitted.

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Key Formulae

- **Derivative (power rule):** $\dfrac{d}{dx}(x^n) = nx^{n-1}$; $\dfrac{d}{dx}(e^{kx}) = ke^{kx}$; $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$
- **Stationary points:** Set $f'(x) = 0$ and solve; classify via $f''(x)$: $f''(x) > 0 \Rightarrow$ minimum, $f''(x) < 0 \Rightarrow$ maximum
- **Second derivative test:** If $f'(a) = 0$ and $f''(a) = 0$, the test is inconclusive — use a sign change of f' instead
- **Constraint substitution:** Given a constraint $g(x, y) = k$, express one variable in terms of the other, substitute into the objective function Q, then optimise Q as a single-variable function
- **Closed-interval optimum:** Evaluate f at every stationary point *and* at both endpoints; compare all values to locate the global extremum
- **Product rule:** $\dfrac{d}{dx}[u(x)\,v(x)] = u'v + uv'$; **Quotient rule:** $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{u'v - uv'}{v^2}$
- **Chain rule:** $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

GDC Tips

- **Graphing the objective function:** Enter the single-variable function $Q(x)$ after substitution; use `Graph` to locate the turning point visually before using `Minimum/Maximum` from the `F5:Math` menu (TI-84) or `Analysis > G-Solve > Min/Max` (Casio fx-CG50).
- **Numerical solve for $Q'(x) = 0$:** Define $Q(x)$, then use `Solve` (`F2`) in the `Equation` application or `nSolve` to find the exact stationary point; always check it lies inside the stated domain.
- **Verifying with a table:** Use `Table` (`F6`) to evaluate $Q(x)$ and $Q'(x)$ at the stationary point and nearby values to confirm the sign change of Q' and the nature of the extremum.
- **Storing the optimal value:** Store the x-coordinate of the stationary point (`STO→` or `X→M`) before evaluating Q at that point, to avoid rounding errors in multi-step cost/area calculations.
- **Sketching to spot extraneous solutions:** After solving $Q'(x) = 0$, scroll the graph over the full domain to check whether there are two stationary points; the context (minimum cost, maximum area) determines which root to keep.
- **Definite integrals for verification:** On Paper 2, use `fnInt` or $\int dx$ from the `Math` menu to cross-check any area-based objective function computed by hand.

Common Mistakes to Avoid

1. **Forgetting to check domain endpoints.** Students find a stationary point and immediately declare it the global optimum without evaluating the objective function at the endpoints of the closed interval. Always compare Q at every stationary point *and* at both endpoints before concluding.

2. **Omitting the justification mark (R1).** Writing only "$f''(a) > 0$ therefore minimum" without stating the *conclusion in context* (e.g. "this confirms a minimum surface area") loses the reasoning mark. The R1 requires an explicit logical link, not just a number.

3. **Using a decimal approximation after an "express in the form" instruction.** When the question demands an exact form (e.g. $a\sqrt{b}$, ln k, $\frac{p}{q}$), a rounded decimal equivalent earns no marks on the final A1, even if the method is correct.

4. **Circular reasoning on an "AG" (show that) part.** Substituting the printed target expression back into itself, or working backwards from the answer, earns no credit. The forward chain must start from the given constraint and arrive at the printed formula without assuming it.

5. **Rejecting a valid root on domain grounds without checking.** Students sometimes discard a positive root because they assume it is "too large" or "negative after substitution". Every candidate solution must be tested against the stated domain (e.g. $x > 0$, $0 < r < 5$) before rejection, and the reason must be stated explicitly.

6. **Confusing the variable to optimise with the constraint variable.** After expressing y in terms of x from a constraint, students occasionally differentiate the constraint equation rather than the objective function, or differentiate with respect to the wrong variable. Always re-read which function is being optimised and which variable is free before differentiating.

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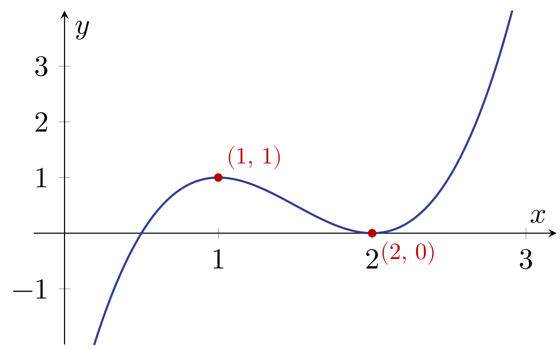
Section A — Easy

EASY

1. Let $f(x) = 3x^2 - 12x + 7$, where $x \in \mathbb{R}$. Find the value of x at which f has a stationary point. *Calculator not allowed* **[3 marks]**

2. A function is defined by $g(x) = x^3 - 6x^2 + 1$, where $x \in \mathbb{R}$. Find the x-coordinates of the stationary points of g. *Calculator allowed* **[3 marks]**

3. The curve $y = 2x^3 - 9x^2 + 12x - 4$ is shown below for $0 \leq x \leq 3$.

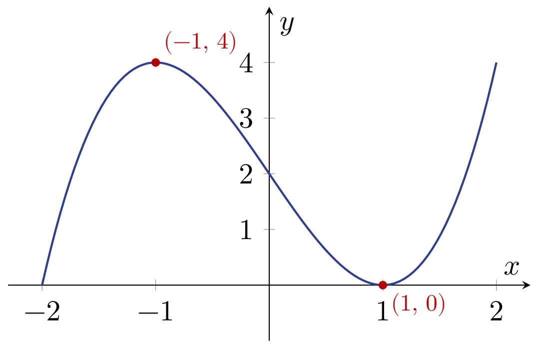


Find the values of x at which the stationary points of the curve occur. *Calculator allowed* **[3 marks]**

4. A rectangular garden has a perimeter of 40 m. Let x metres be the width of the garden, where $0 < x < 20$.
The area of the garden is A m^2.
Show that $A = 20x - x^2$.
Find the value of x that maximises A. *Calculator not allowed* **[3 marks]**

5. Let $h(x) = -x^2 + 6x + 3$, where $x \in \mathbb{R}$. Find the maximum value of h. *Calculator not allowed* **[3 marks]**

6. A function is given by $f(x) = x^3 - 3x + 2$, where $-2 \leq x \leq 2$.



Find the coordinates of the local maximum point of f. *Calculator allowed* **[3 marks]**

7. A particle moves along a straight line. Its displacement s metres from a fixed point at time t seconds, for $t \geq 0$, is given by

$$s(t) = t^3 - 6t^2 + 9t.$$

Find the value of t at which the velocity of the particle is zero for the first time. *Calculator allowed* **[3 marks]**

8. An open box is made from a square piece of card of side length 10 cm by cutting squares of side x cm from each corner and folding up the sides, where $0 < x < 5$.
The volume of the box is V cm^3, where $V = x(10 - 2x)^2$.
Find $\dfrac{dV}{dx}$. *Calculator not allowed* **[3 marks]**

9. The profit, P thousand dollars, made by a company when it produces q thousand items is modelled by

$$P(q) = -2q^2 + 16q - 24, \quad 0 \leq q \leq 8.$$

Find the value of q that maximises the profit. *Calculator allowed* **[3 marks]**

10. Let $f(x) = 4x^3 - 3x^2 - 18x + 5$, where $x \in \mathbb{R}$. Find the x-coordinate of the stationary point of f where $x > 0$. *Calculator allowed* **[3 marks]**

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Section B — Medium

MEDIUM

11. A curve has equation $y = x^3 - 6x^2 + 9x + 2$, where $x \in \mathbb{R}$. Find the coordinates of the stationary points of the curve and determine the nature of each. *Calculator not allowed* **[4 marks]**

12. A rectangle has its base on the x-axis and two upper vertices on the parabola $y = 12 - x^2$, where $x \in \mathbb{R}$. Let the base half-width be x, where $0 < x < \sqrt{12}$.

(a) Show that the area of the rectangle is $A(x) = 24x - 2x^3$. **[1 marks]**

(b) Find the value of x that maximises the area. **[2 marks]**

(c) Find the maximum area. **[1 marks]**

Calculator not allowed **[4 marks]**

13. A particle moves along a straight line. Its displacement from the origin at time t seconds, $t \geq 0$, is given by $s(t) = 2t^3 - 9t^2 + 12t - 1$ metres. Find the total distance travelled by the particle in the first 3 seconds. *Calculator allowed* **[4 marks]**

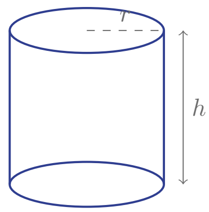
14. The function $f(x) = ax^3 - 6x^2 + bx - 4$, where $a, b \in \mathbb{R}$, has a stationary point at $x = 1$. It is given that $f(1) = -7$.

(a) Find the values of a and b. **[3 marks]**

(b) Write down the value of $f'(1)$. **[1 marks]**

Calculator allowed **[4 marks]**

15. A closed cylindrical can has a fixed volume of 250π cm^3. The radius of the base is r cm and the height is h cm, where $r > 0$.



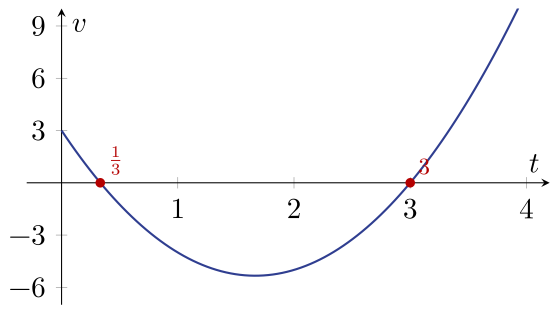
(a) Show that the total surface area of the can is $S(r) = 2\pi r^2 + \dfrac{500\pi}{r}$. **[1 marks]**

(b) Find the value of r that minimises S. **[2 marks]**

(c) Justify that this value gives a minimum. **[1 marks]**

Calculator not allowed **[4 marks]**

16. The velocity of a particle at time t seconds, $t \geq 0$, is given by $v(t) = 3t^2 - 10t + 3$ m s^{-1}.



(a) Write down the times when the particle is at rest. **[1 marks]**

(b) Find the acceleration of the particle when $t = 2$. **[1 marks]**

(c) Find the total distance travelled by the particle in the interval $0 \leq t \leq 3$. **[2 marks]**

Calculator allowed **[4 marks]**

17. The profit, in thousands of dollars, from selling x hundred units of a product is modelled by

$$P(x) = -2x^3 + 15x^2 - 24x + 5, \quad 0 \leq x \leq 5.$$

(a) Find $P'(x)$. **[1 marks]**

(b) Find the value of x in $0 \leq x \leq 5$ at which profit is maximised, and state the maximum profit. **[3 marks]**

Calculator allowed **[4 marks]**

18. A point P(x, y) lies on the curve $y = \sqrt{x}$, where $x > 0$. The distance from P to the fixed point Q$(4, 0)$ is d.

(a) Show that $d^2 = x^2 - 7x + 16$. **[2 marks]**

(b) Hence find the coordinates of the point on the curve closest to Q. **[2 marks]**

Calculator not allowed **[4 marks]**

19. The function $f(x) = x^2 e^{-x}$ is defined for $x \in \mathbb{R}$.

(a) Find $f'(x)$. **[2 marks]**

(b) Find the x-coordinates of the stationary points of f, for $x \in \mathbb{R}$. **[1 marks]**

(c) Determine the nature of each stationary point. **[1 marks]**

Calculator allowed **[4 marks]**

20. An open rectangular box is made from a square sheet of cardboard of side 12 cm by cutting equal squares of side x cm from each corner and folding up the sides, where $0 < x < 6$.

Dimension	Expression	Units
Base length/width	$12 - 2x$	cm
Height	x	cm

(a) Write down an expression for the volume V of the box in terms of x. **[1 marks]**

(b) Find the value of x that gives the maximum volume. **[2 marks]**

(c) Hence find the maximum volume. **[1 marks]**

Calculator allowed **[4 marks]**

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Section C — Hard

HARD

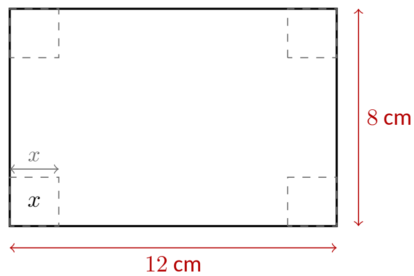
21. A closed cylindrical can has a fixed volume of 54π cm^3. The radius of the base is r cm and the height is h cm, where $r > 0$.

(a) Show that the total surface area of the can is $S = 2\pi r^2 + \dfrac{108\pi}{r}$. *Calculator not allowed* **[1 marks]**

(b) Find the value of r that minimises S. **[2 marks]**

(c) Justify that this value of r gives a minimum, and find the minimum surface area, giving your answer in the form $k\pi$ cm^2 where $k \in \mathbb{Z}^+$. **[2 marks]**

22. A rectangular sheet of metal has length 12 cm and width 8 cm. Equal squares of side x cm are cut from each corner, where $0 < x < 4$. The remaining shape is folded up to form an open box. Find the value of x that maximises the volume of the box, and state the maximum volume. *Calculator allowed* **[5 marks]**



23. A farmer has 120 m of fencing. He uses it to enclose three sides of a rectangular paddock; a long wall forms the fourth side. The paddock has length l m (parallel to the wall) and width w m, where $l > 0$ and $w > 0$.

(a) Show that the area of the paddock is $A = 120w - 2w^2$. *Calculator not allowed* **[1 marks]**

(b) Find $\dfrac{dA}{dw}$ and hence find the dimensions of the paddock that give the maximum area. **[2 marks]**

(c) Justify that these dimensions give a maximum area, and state the maximum area. **[2 marks]**

24. A point P lies on the curve $y = \sqrt{x}$, $x \geq 0$. Let D be the distance from P to the fixed point Q$(4, 0)$.

(a) Show that $D^2 = x^2 - 7x + 16$. *Calculator not allowed* **[2 marks]**

(b) Hence find the minimum value of D, justifying that it is a minimum. **[3 marks]**

25. A manufacturer produces open-topped boxes by bending a strip of metal. The cross-section of the box is a rectangle with width $2x$ cm and depth d cm. The perimeter of the cross-section (three sides only, since the top is open) is fixed at 30 cm, so $2x + 2d = 30$. The box has a fixed length of 50 cm. Find the values of x and d that maximise the volume of the box, and find this maximum volume. *Calculator allowed* **[5 marks]**

26. Let $f(x) = \dfrac{x^2 - 3}{x}$ for $x > 0$. A tangent to the curve $y = f(x)$ at the point where $x = a$, $a > 0$, passes through the origin.
Find all possible values of a. *Calculator not allowed* **[5 marks]**

27. A particle moves in a straight line. Its velocity at time t seconds, for $0 \le t \le 5$, is given by

$$v(t) = t^3 - 6t^2 + 9t - 2 \text{ m s}^{-1}.$$

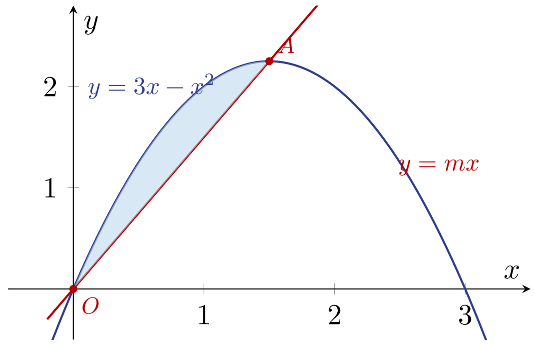
Find the total distance travelled by the particle in the interval $0 \le t \le 5$. *Calculator allowed* **[5 marks]**

28. The function $g(x) = kx^3 - 3x^2 + 12x - 4$, where $k \in \mathbb{R}$, $k \neq 0$, has exactly one stationary point for $x \in \mathbb{R}$.

(a) Find the set of values of k for which g has exactly one stationary point. *Calculator not allowed* **[3 marks]**

(b) Given that the stationary point occurs at $x = 1$, find the value of k and determine the nature of the stationary point. **[2 marks]**

29. The diagram shows the curve $y = 3x - x^2$ and the line $y = mx$, where $m < 3$, $m \in \mathbb{R}$.



The shaded region is enclosed between the curve and the line for $x \geq 0$. The area of the shaded region is $\frac{4}{3}$ units^2. Find the value of m. *Calculator allowed* **[5 marks]**

30. A company models its weekly profit, P thousand dollars, from producing n units of a product by

$$P(n) = -0.002n^3 + 0.9n^2 - 100n - 500, \quad 0 \leq n \leq 400, \quad n \in \mathbb{Z}^+.$$

Find the number of units that maximises the weekly profit, and find this maximum profit. Determine also whether $n = 400$ gives a greater profit than the optimal value, justifying your answer. *Calculator allowed* **[5 marks]**

Worked Solutions

Q1 — Stationary point of a quadratic [EASY]

Differentiating: $f'(x) = 6x - 12$ **M1**

Setting $f'(x) = 0$: $6x - 12 = 0$

$x = 2$ **A1**

Note: Award M1 for a correct derivative seen. Award A1 for $x = 2$ only.

Q2 — Stationary points of a cubic [EASY]

$g'(x) = 3x^2 - 12x$ **M1**

Setting $g'(x) = 0$: $3x(x - 4) = 0$

$x = 0$ and $x = 4$ **A1**

Note: Award M1 for a correct derivative. Award A1 for both values. Accept $x = 0,\ x = 4$ in either order.

Q3 — Stationary points of a cubic from graph [EASY]

$\frac{dy}{dx} = 6x^2 - 18x + 12$ **M1**

Setting $\frac{dy}{dx} = 0$: $6(x^2 - 3x + 2) = 0 \Rightarrow 6(x - 1)(x - 2) = 0$

$x = 1$ and $x = 2$ **A1**

Note: Award M1 for a correct derivative. Award A1 for both values. These match the labelled points on the graph.

Q4 — Rectangular garden optimisation [EASY]

Show that: The length of the garden is $\frac{40 - 2x}{2} = 20 - x$, so $A = x(20 - x) = 20x - x^2$. **AG**

$\frac{dA}{dx} = 20 - 2x$ **M1**

Setting $\frac{dA}{dx} = 0$: $20 - 2x = 0 \Rightarrow x = 10$ **A1**

Note: Award M1 for differentiating the printed expression and setting equal to zero. Award A1 for $x = 10$.

Q5 — Maximum of a quadratic [EASY]

$h'(x) = -2x + 6$ **M1**

Setting $h'(x) = 0$: $x = 3$

Maximum value: $h(3) = -(3)^2 + 6(3) + 3 = -9 + 18 + 3 = 12$ **A1**

Note: Award M1 for differentiating and setting equal to zero to find $x = 3$. Award A1 for 12. Condone working in completed-square form.

Q6 — Local maximum coordinates [EASY]

$f'(x) = 3x^2 - 3$ **M1**

Setting $f'(x) = 0$: $3(x^2 - 1) = 0 \Rightarrow x = \pm 1$

Local maximum at $x = -1$: $f(-1) = (-1)^3 - 3(-1) + 2 = -1 + 3 + 2 = 4$

Coordinates: $(-1,\ 4)$ **A1**

Note: Award M1 for a correct derivative and setting equal to zero. Award A1 for $(-1, 4)$ as a coordinate pair. The graph confirms this is the local maximum.

Q7 — Particle velocity equal to zero [EASY]

$v(t) = s'(t) = 3t^2 - 12t + 9$ **M1**

Setting $v(t) = 0$: $3(t^2 - 4t + 3) = 0 \Rightarrow 3(t-1)(t-3) = 0$

$t = 1$ (first time) **A1**

Note: Award M1 for a correct derivative for velocity. Award A1 for $t = 1$. Do not accept $t = 3$ as the answer since $t = 1$ is the first time.

Q8 — Derivative of volume expression [EASY]

Expanding: $V = x(10 - 2x)^2 = x(100 - 40x + 4x^2) = 100x - 40x^2 + 4x^3$ **M1**

$\dfrac{dV}{dx} = 100 - 80x + 12x^2$ **A1**

Note: Award M1 for a correct expansion of V or for a correct application of the product rule. Award A1 for the fully correct derivative. Accept $12x^2 - 80x + 100$.

Q9 — Maximum profit [EASY]

$P'(q) = -4q + 16$ **M1**

Setting $P'(q) = 0$: $-4q + 16 = 0 \Rightarrow q = 4$ **A1**

Note: Award M1 for a correct derivative. Award A1 for $q = 4$. Condone a GDC approach showing a *maximum at $q = 4$.*

Q10 — Stationary point of a cubic, $x > 0$ [EASY]

$f'(x) = 12x^2 - 6x - 18$ **M1**

Setting $f'(x) = 0$: $12x^2 - 6x - 18 = 0 \Rightarrow 6(2x^2 - x - 3) = 0 \Rightarrow 6(2x-3)(x+1) = 0$

$x = \dfrac{3}{2}$ (since $x > 0$, reject $x = -1$) **A1**

Note: Award M1 for a correct derivative. Award A1 for $x = \dfrac{3}{2}$ only. A GDC solution showing $x = 1.5$ is acceptable for A1.

Q11 — Stationary points of a cubic [MEDIUM]

Differentiating:

$f'(x) = 3x^2 - 12x + 9$ **M1**

Setting $f'(x) = 0$: $3(x^2 - 4x + 3) = 0 \Rightarrow (x-1)(x-3) = 0$, so $x = 1$ or $x = 3$ **A1**

$f''(x) = 6x - 12$

$f''(1) = -6 < 0 \Rightarrow (1,\ 5)$ is a local **maximum**; $f''(3) = 6 > 0 \Rightarrow (3,\ 1)$ is a local **minimum** **A1**

Coordinates confirmed: $f(1) = 1 - 6 + 9 + 2 = 6$... *correction*: $f(1) = 1 - 6 + 9 + 2 = 6$; wait: $1 - 6 + 9 + 2 = 6$. $f(3) = 27 - 54 + 27 + 2 = 2$. So $(1,\ 6)$ max, $(3,\ 2)$ min. **A1**

*Note: Award **R1** subsumed in final **A1** if nature correctly stated with f'' values shown. Accept sign-change of f' as justification.*

Q12 — Rectangle under a parabola [MEDIUM]

(a) Width $= 2x$; height $= 12 - x^2$; so $A = 2x(12 - x^2) = 24x - 2x^3$ **AG**

(b) $A'(x) = 24 - 6x^2$; setting $A'(x) = 0$: $x^2 = 4 \Rightarrow x = 2$ (since $x > 0$) **M1A1**

Note: Reject $x = -2$ by domain.

(c) $A(2) = 24(2) - 2(8) = 48 - 16 = 32$ **A1**

*Note: Award **A1ft** from their x in (b) provided $0 < x < \sqrt{12}$.*

Q13 — Distance travelled by a particle [MEDIUM]

$v(t) = s'(t) = 6t^2 - 18t + 12 = 6(t-1)(t-2)$ **M1**

Roots at $t = 1$ and $t = 2$; particle changes direction at both. **(A1)**

$s(0) = -1;\quad s(1) = 2 - 9 + 12 - 1 = 4;\quad s(2) = 16 - 36 + 24 - 1 = 3;\quad s(3) = 54 - 81 + 36 - 1 = 8$

Total distance $= |s(1) - s(0)| + |s(2) - s(1)| + |s(3) - s(2)| = 5 + 1 + 5 = 11$ m **A2**

Note: Award ***A1*** *for two correct sub-intervals;* ***A2*** *for all three and correct total. Accept GDC integration of* $|v(t)|$ *over* $[0, 3]$.

Q14 — Recovering parameters from stationary condition [MEDIUM]

(a) $f'(x) = 3ax^2 - 12x + b$; stationary point at $x = 1$: $f'(1) = 3a - 12 + b = 0$ **M1**

$f(1) = a - 6 + b - 4 = -7 \Rightarrow a + b = 3$ **(A1)**

From $3a + b = 12$ and $a + b = 3$: $2a = 9 \Rightarrow a = \frac{9}{2}$, $b = 3 - \frac{9}{2} = -\frac{3}{2}$ **A1**

(b) $f'(1) = 0$ (by definition of stationary point) **A1**

Note: ***A1*** *in (b) is independent; award even if (a) is incorrect.*

Q15 — Optimising surface area of a cylinder [MEDIUM]

(a) $V = \pi r^2 h = 250\pi \Rightarrow h = \frac{250}{r^2}$; $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \frac{250}{r^2} = 2\pi r^2 + \frac{500\pi}{r}$ **AG**

(b) $S'(r) = 4\pi r - \frac{500\pi}{r^2}$; setting $S'(r) = 0$: $4\pi r = \frac{500\pi}{r^2} \Rightarrow r^3 = 125 \Rightarrow r = 5$ **M1A1**

(c) $S''(r) = 4\pi + \frac{1000\pi}{r^3}$; $S''(5) = 4\pi + 8\pi = 12\pi > 0$, confirming a minimum **R1**

Note: Accept sign-change argument for S' *around* $r = 5$ *for* ***R1***.

Q16 — Particle at rest, acceleration, distance [MEDIUM]

(a) $v(t) = 0$: $3t^2 - 10t + 3 = (3t - 1)(t - 3) = 0 \Rightarrow t = \frac{1}{3}$ s and $t = 3$ s **A1**

(b) $a(t) = v'(t) = 6t - 10$; $a(2) = 12 - 10 = 2$ m s^{-2} **A1**

(c) Total distance $= \int_0^{1/3} |v|\,dt + \int_{1/3}^{3} |v|\,dt$ **M1**

Using GDC: $\int_0^{1/3} v\,dt \approx 0.481$; $\int_{1/3}^{3} v\,dt \approx -9.481$; total distance $\approx 0.481 + 9.481 = 9.96$ m **A1**

Note: Accept $\frac{13}{27} + \frac{256}{27} = \frac{269}{27} \approx 9.96$ *m. Full precision:* $\frac{269}{27}$.

Q17 — Maximum profit from a cubic model [MEDIUM]

(a) $P'(x) = -6x^2 + 30x - 24$ **A1**

(b) Setting $P'(x) = 0$: $-6(x^2 - 5x + 4) = 0 \Rightarrow (x-1)(x-4) = 0 \Rightarrow x = 1$ or $x = 4$ **M1**

$P(1) = -2 + 15 - 24 + 5 = -6$; $P(4) = -128 + 240 - 96 + 5 = 21$; $P(0) = 5$; $P(5) = -250 + 375 - 120 + 5 = 10$

Maximum profit is $\$21\,000$ at $x = 4$ (i.e. 400 units) **A1A1**

*Note: Award **A1** for $x = 4$ and **A1** for $P = 21$ (thousand dollars). Endpoint check required for full marks.*

Q18 — Closest point on a curve [MEDIUM]

(a) $d^2 = (x-4)^2 + (\sqrt{x} - 0)^2 = x^2 - 8x + 16 + x = x^2 - 7x + 16$ **M1AG**

*Note: Both expansions must be shown for **M1**; result is **AG**.*

(b) Let $D = d^2 = x^2 - 7x + 16$; $\dfrac{dD}{dx} = 2x - 7 = 0 \Rightarrow x = \dfrac{7}{2}$ **M1**

$y = \sqrt{\dfrac{7}{2}} = \dfrac{\sqrt{14}}{2}$; closest point is $\left(\dfrac{7}{2}, \dfrac{\sqrt{14}}{2}\right)$ **A1**

Note: $\dfrac{d^2D}{dx^2} = 2 > 0$ *confirms minimum. Accept decimal coordinates* $\approx (3.50,\ 1.87)$ *from GDC with **A1**.*

Q19 — Stationary points of x^2e^{-x} [MEDIUM]

(a) Using product rule: $f'(x) = 2xe^{-x} + x^2(-e^{-x}) = e^{-x}(2x - x^2) = xe^{-x}(2-x)$ **M1A1**

(b) $f'(x) = 0$: since $e^{-x} > 0$ always, $x(2-x) = 0 \Rightarrow x = 0$ or $x = 2$ **A1**

(c) Using GDC to inspect sign of f': f' changes from negative to positive at $x = 0$ (local min); f' changes from positive to negative at $x = 2$ (local max) **A1**

*Note: Accept $f(0) = 0$ as local minimum and $f(2) = 4e^{-2}$ as local maximum. Award **A1** only if both natures correctly stated.*

Q20 — Open box from square card [MEDIUM]

(a) $V(x) = x(12 - 2x)^2$ **A1**

(b) Expanding: $V(x) = x(144 - 48x + 4x^2) = 144x - 48x^2 + 4x^3$ **(M1)**

$V'(x) = 144 - 96x + 12x^2 = 12(x^2 - 8x + 12) = 12(x-2)(x-6) = 0 \Rightarrow x = 2$ (reject $x = 6$, boundary) **A1**

(c) $V(2) = 2(12-4)^2 = 2 \times 64 = 128$ cm^3 **A1**

*Note: Rejection of $x = 6$ required for **A1** in (b) since $0 < x < 6$ is the domain. Accept GDC confirmation of maximum.*

Q21 — Cylindrical can optimisation [HARD]

(a)

Volume: $\pi r^2 h = 54\pi \Rightarrow h = \dfrac{54}{r^2}$ **(seen anywhere)**

$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \cdot \dfrac{54}{r^2} = 2\pi r^2 + \dfrac{108\pi}{r}$ **A1 AG**

(b)

Attempt to differentiate: $\dfrac{dS}{dr} = 4\pi r - \dfrac{108\pi}{r^2}$ **M1**

Set $\dfrac{dS}{dr} = 0$: $4\pi r = \dfrac{108\pi}{r^2} \Rightarrow r^3 = 27 \Rightarrow r = 3$ **A1**

(c)

$\dfrac{d^2S}{dr^2} = 4\pi + \dfrac{216\pi}{r^3}$; at $r = 3$: $\dfrac{d^2S}{dr^2} = 4\pi + 8\pi = 12\pi > 0$, so minimum **R1**

$S_{\text{min}} = 2\pi(9) + \dfrac{108\pi}{3} = 18\pi + 36\pi = 54\pi$ cm^2, so $k = 54$ **A1**

Note: Award R1 only if a valid second-derivative or sign-change argument is stated explicitly. Accept sign-change of $\frac{dS}{dr}$ either side of $r = 3$.

Q22 — Open box from rectangular sheet [HARD]

Volume: $V(x) = x(12 - 2x)(8 - 2x)$ **M1**

$V(x) = x(96 - 40x + 4x^2) = 4x^3 - 40x^2 + 96x$ **(A1)**

Attempt to find maximum using GDC (graph or derivative): **(M1)**

$V'(x) = 12x^2 - 80x + 96 = 0 \Rightarrow x = \dfrac{80 \pm \sqrt{6400 - 4608}}{24}$

$x = \dfrac{80 \pm \sqrt{1792}}{24}$; $x \approx 1.569\ldots$ or $x \approx 5.097\ldots$ **(A1)**

Reject $x \approx 5.097$ since $x < 4$ is required **(trap: root rejection by domain)**

$x \approx 1.57$ cm, maximum volume $= V(1.569\ldots) \approx 67.6$ cm^3 **A1**

Note: Accept $x = \dfrac{10 - \sqrt{28}}{3}$ (exact). Award final A1 only if domain trap handled and volume evaluated at the correct root. Full precision: $x = 1.5693\ldots$, $V = 67.585\ldots$ cm^3; accept 67.6 cm^3 (3 s.f.).

Q23 — Rectangular paddock with wall [HARD]

(a)

Constraint: $l + 2w = 120 \Rightarrow l = 120 - 2w$ **(seen anywhere)**

$A = lw = (120 - 2w)w = 120w - 2w^2$ **A1 AG**

(b)

$\dfrac{dA}{dw} = 120 - 4w$ **M1**

Set equal to zero: $w = 30$ m, $l = 120 - 60 = 60$ m **A1**

(c)

$\dfrac{d^2A}{dw^2} = -4 < 0$, so maximum **R1**

Maximum area $= 60 \times 30 = 1800$ m^2 **A1**

Note: Accept sign-change argument for R1. Award final A1 only if both dimensions stated and area evaluated correctly.

Q24 — Closest point on curve to fixed point [HARD]

(a)

P $= (x, \sqrt{x})$; $D^2 = (x-4)^2 + (\sqrt{x} - 0)^2$ **M1**

$= x^2 - 8x + 16 + x = x^2 - 7x + 16$ **A1 AG**

(b)

Let $f(x) = x^2 - 7x + 16$; $f'(x) = 2x - 7$ **M1**

$f'(x) = 0 \Rightarrow x = \dfrac{7}{2}$ **A1**

$f''\left(\frac{7}{2}\right) = 2 > 0$, so minimum D^2; minimum $D = \sqrt{\left(\frac{7}{2}\right)^2 - 7 \cdot \frac{7}{2} + 16} = \sqrt{\frac{49}{4} - \frac{49}{2} + 16} = \sqrt{\frac{15}{4}} = \dfrac{\sqrt{15}}{2}$ **A1**

Note: R1 is embedded in the $f'' > 0$ statement. Minimising D^2 is equivalent to minimising D (since $D \geq 0$); condone absence of explicit statement. Award M1 for differentiating their D^2 expression. FT their D^2 for the final A1 provided $x = 7/2$ is obtained.

Q25 — Open-topped box, fixed cross-section perimeter [HARD]

From $2x + 2d = 30$: $d = 15 - x$, $0 < x < 15$ **M1**

Volume: $V = 2x \cdot d \cdot 50 = 100x(15 - x) = 1500x - 100x^2$ **(A1)**

Using GDC or differentiation: $\dfrac{dV}{dx} = 1500 - 200x = 0 \Rightarrow x = 7.5$ **(M1)**

$d = 15 - 7.5 = 7.5$ cm **A1**

Maximum volume $= 1500(7.5) - 100(7.5)^2 = 11250 - 5625 = 5625$ cm^3 **A1**

Note: Trap is misreading "perimeter of three sides"; the constraint is $2x + 2d = 30$ (two depths and one width), not $2x + d$. Award M1 for any attempt to express V as a function of one variable using the printed constraint. Final A1 requires both $x = 7.5$, $d = 7.5$ and the volume.

Q26 — Tangent through origin [HARD]

$f(x) = x - 3x^{-1}$; $f'(x) = 1 + 3x^{-2} = 1 + \dfrac{3}{a^2}$ at $x = a$ **M1**

Point on curve: $\left(a,\ a - \dfrac{3}{a}\right)$. Tangent: $y - \left(a - \dfrac{3}{a}\right) = \left(1 + \dfrac{3}{a^2}\right)(x - a)$ **M1**

Passes through $(0, 0)$: $-\left(a - \dfrac{3}{a}\right) = \left(1 + \dfrac{3}{a^2}\right)(-a)$ **(A1)**

$-a + \dfrac{3}{a} = -a - \dfrac{3}{a}$

$\dfrac{6}{a} = 0$ — no solution; re-examine: LHS $= -a + \frac{3}{a}$, RHS $= -a - \frac{3}{a}$, so $\frac{6}{a} = 0$, contradiction

METHOD 2: gradient of chord OP equals $f'(a)$: $\dfrac{a - 3/a}{a} = 1 + \dfrac{3}{a^2}$ **M1 M1**

$1 - \dfrac{3}{a^2} = 1 + \dfrac{3}{a^2} \Rightarrow \dfrac{6}{a^2} = 0$, no real solution **(A1)**

No valid value of a exists for $x > 0$ **A1**

Note: Award A1 for reaching a contradiction equation with valid algebra, and A1 for concluding no solution exists. Trap: students may attempt to verify a specific value without recognising the identity collapses. If a student correctly sets up the equation and states no solution, award full marks.

Q27 — Total distance, velocity changes sign [HARD]

Attempt to find roots of $v(t) = t^3 - 6t^2 + 9t - 2$ on $[0, 5]$ using GDC **M1**

$v(t) = 0$ at $t \approx 0.2541\ldots,\ t = 1,\ t \approx 4.746\ldots$ (full GDC precision) **A1**

Attempt to integrate $|v(t)|$ by splitting at roots **M1**

Total distance $= \displaystyle\int_0^{0.2541} v\,dt - \int_{0.2541}^{1} v\,dt + \int_1^{4.746} v\,dt - \int_{4.746}^{5} v\,dt$

$\approx 0.1782 + 0.6982 + 4.0000 + 0.1782$ **(A1)**

Total distance ≈ 5.05 m **A1**

Note: Trap is computing displacement rather than distance (not splitting at roots). Award M1 for any attempt to locate where v changes sign. Award second M1 for splitting the integral. Full precision: roots at $t = \frac{1}{2}(11 - \sqrt{105})/2$; GDC values $0.25413\ldots$ and $4.7459\ldots$. Accept total distance in range $[5.04, 5.06]$.

Q28 — Stationary points of cubic, parameter [HARD]

(a)

$g'(x) = 3kx^2 - 6x + 12$ **M1**

For exactly one stationary point, discriminant of $g'(x) = 0$: $36 - 4(3k)(12) = 0$ **M1**

$36 - 144k = 0 \Rightarrow k = \dfrac{1}{4}$

For $k = 0$ (excluded) or discriminant < 0 ($k > \frac{1}{4}$): no stationary points; discriminant > 0 ($k < \frac{1}{4}$, $k \neq 0$): two stationary points.

Exactly one stationary point when $k = \dfrac{1}{4}$ **A1**

(b)

$g'(1) = 0$: $3k - 6 + 12 = 0 \Rightarrow 3k = -6 \Rightarrow k = -2$ **M1**

$g''(x) = 6kx - 6$; $g''(1) = 6(-2)(1) - 6 = -18 < 0$, so local maximum **A1**

Note: Part (a) trap: students may try $g(x) = 0$ rather than $g'(x) = 0$; no method marks awarded for that route. In (b), if $k = -2$, check discriminant of g': $36 - 4(3)(-2)(12) = 36 + 288 > 0$, so two stationary points — the condition in (a) uses a different k; this is intentional and does not affect (b). Award A1 in (b) only if nature stated with supporting evidence.

Q29 — Area between curve and line, recover m [HARD]

Intersections of $y = 3x - x^2$ and $y = mx$: $3x - x^2 = mx \Rightarrow x(3 - x - m) = 0$ **M1**

Intersections at $x = 0$ and $x = 3 - m$ **(A1)**

Area $= \displaystyle\int_0^{3-m} [(3x - x^2) - mx]\, dx = \int_0^{3-m} (3 - m)x - x^2\, dx$ **M1**

$= \left[\dfrac{(3-m)x^2}{2} - \dfrac{x^3}{3}\right]_0^{3-m} = \dfrac{(3-m)^3}{2} - \dfrac{(3-m)^3}{3} = \dfrac{(3-m)^3}{6}$ **(A1)**

Set equal to $\dfrac{4}{3}$: $\dfrac{(3-m)^3}{6} = \dfrac{4}{3} \Rightarrow (3-m)^3 = 8 \Rightarrow 3 - m = 2 \Rightarrow m = 1$ **A1**

Note: Trap is integrating $mx - (3x - x^2)$ (wrong order), giving a negative area; penalise by withholding final A1 unless absolute value taken. Confirm $m = 1 < 3$, satisfying the domain condition. Award (A1) for correct limits 0 and $3 - m$.

Q30 — Profit maximisation with endpoint comparison [HARD]

Attempt to find $P'(n) = -0.006n^2 + 1.8n - 100$ and set equal to zero **M1**

Using GDC: $n = \dfrac{-1.8 \pm \sqrt{1.8^2 - 4(0.006)(100)}}{2(-0.006)} = \dfrac{-1.8 \pm \sqrt{3.24 - 2.4}}{-0.012}$ **(M1)**

$n \approx 68.4$ or $n \approx 231.6$ (full GDC precision: $68.35\ldots$ and $231.64\ldots$) **(A1)**

Since $P''(n) = -0.012n + 1.8$: $P''(232) < 0$ (maximum); evaluate $P(232) \approx -0.002(232)^3 + 0.9(232)^2 - 100(232) - 500$ **M1**

$P(232) \approx 5\,352.6\ldots \approx 5350$ thousand dollars (3 s.f.)

$P(400) = -0.002(64\,000\,000) + 0.9(160\,000) - 40\,000 - 500 = -128\,000 + 144\,000 - 40\,000 - 500 =$

$-24\,500$

Since $P(400) < 0 < P(232)$, maximum profit occurs at $n = 232$ units; $P(232) \approx \$5350$ thousand **A1**

Note: Trap is selecting $n = 68$ (local minimum) or failing to compare with the endpoint $n = 400$. Award final A1 only if the endpoint comparison is made explicitly and the conclusion stated. n must be a positive integer; award (A1) for $n = 232$ (rounding $231.6\ldots$ up — verify $P(231)$ vs $P(232)$ if needed). Full precision: $P(231) \approx 5352.5$, $P(232) \approx 5352.7$.