IB Math AI SL — Topic 2

Further Functions & Graphs

Functions & Inverses · Key Features · Quadratic, Cubic, Exponential & Sinusoidal Graphs

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

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Name:________________________________ Date:________________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. A graphic display calculator is required throughout; support GDC answers with the values entered and the feature used.

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Key Formulae

- Vertex form of a quadratic: $f(x) = a(x-h)^2 + k$, vertex at (h, k); axis of symmetry $x = h = -\dfrac{b}{2a}$.
- Exponential model: $f(t) = ab^t + c$; y-intercept $= a + c$; horizontal asymptote $y = c$ (never attained).
- Sinusoidal model (degrees): $f(x) = a\sin(bx) + d$; amplitude $= |a|$; period $= \dfrac{360^\circ}{b}$; midline $y = d$.
- Inverse function: $f^{-1}(k)$ is found by solving $f(x) = k$; the graph of f^{-1} is the reflection of f in $y = x$.
- Axis of symmetry (quadratic): $x = -\dfrac{b}{2a}$; the second root satisfies $x_2 = 2h - x_1$ where h is the axis.
- Key exponential values: as $t \to \infty$, $ab^t \to 0$ (if $0 < b < 1$) or $\to \infty$ (if $b > 1$); y-intercept always $= f(0) = a + c$.
- Composite and domain: domain of $f \circ g$ requires $g(x)$ to lie inside the domain of f; range of f = domain of f^{-1}.

GDC Tips

- **Graphing a function and reading intercepts:** Y= $\to$ enter expression $\to$ GRAPH $\to$ 2nd TRACE (CALC) $\to$ 2: zero (set left/right bounds, guess) for x-intercepts; 1: value to evaluate $f(k)$.
- **Finding a maximum or minimum on an interval:** GRAPH $\to$ 2nd TRACE $\to$ 3: minimum or 4: maximum $\to$ set left bound, right bound, guess; read off both x and y coordinates.
- **Solving $f(x) = g(x)$ (intersection):** enter both functions in Y1 and Y2 $\to$ GRAPH $\to$ 2nd TRACE $\to$ 5: intersect $\to$ move cursor near each intersection separately and confirm; record all intersection points.
- **Finding an inverse value $f^{-1}(k)$:** in Y1 enter $f(x)$; in Y2 enter the constant $k \to$ 5: intersect; the x-coordinate of the intersection is $f^{-1}(k)$.
- **Sinusoidal regression (checking parameters):** STAT $\to$ CALC $\to$ C: SinReg; confirm the GDC works in degrees mode (MODE $\to$ Degree) before using any trig graph or regression.
- **Sketching and labelling key features:** after graphing, use WINDOW to set an appropriate view showing all intercepts, the vertex/turning point, and the asymptote; zoom 0: ZoomFit can locate the curve quickly.

Common Mistakes to Avoid

1. **Stating the asymptote as a value, not an equation.** The horizontal asymptote of $y = ab^t + c$ must be written as the equation $y = c$, not just "c". Omitting "$y =$" loses the accuracy mark every time.
2. **Claiming the function attains its asymptote.** Because $ab^t \to 0$ but never equals 0, the function approaches $y = c$ but never reaches it. Writing "the temperature eventually equals c" is incorrect; write "approaches" or "tends to".
3. **Using radians instead of degrees for sinusoidal models.** The period formula is $\dfrac{360^\circ}{b}$, not $\dfrac{2\pi}{b}$. Always check the GDC is in Degree mode before graphing or solving any trigonometric equation in this course.
4. **Confusing $f^{-1}(k)$ with $\dfrac{1}{f(k)}$.** The inverse function notation f^{-1} means "the input that produces output k", found by solving $f(x) = k$. It is never the reciprocal $[f(k)]^{-1}$.
5. **Missing the second intersection (or second root) of a quadratic/graph.** When solving $f(x) = g(x)$ or finding x-intercepts, always zoom out and scroll to check for a second (or third) solution. Use the symmetry property $x_2 = 2h - x_1$ as a check for quadratics.
6. **Premature rounding in chained parts.** Rounding an intermediate result (e.g. a vertex coordinate or an intersection point) and using that rounded value in a subsequent calculation causes accumulating error. Store the full unrounded GDC value and round only the final answer to 3 significant figures.

Section A — Further Functions and Graphs EASY (10 questions × 3 marks = 30 marks)

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1. EASY Calculator **[3 marks]**

A function is defined as $f(x) = 3x - 7$ for $x \in \mathbb{R}$.

(a) Write down the value of $f(4)$. [1]

(b) Find the value of x such that $f(x) = 14$. [2]

2. EASY Calculator **[3 marks]**

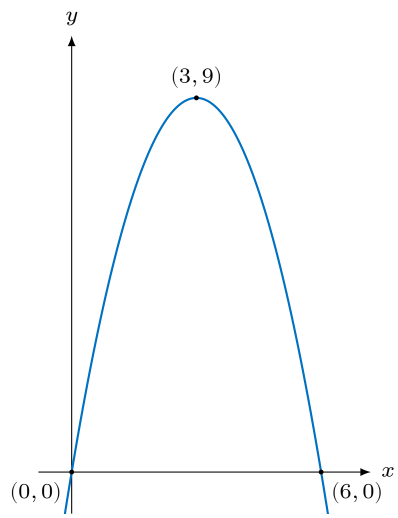
A function is defined as $g(x) = 2x^2 - 8$ for $x \geq 0$.

(a) Write down the value of $g(0)$. [1]

(b) Find $g^{-1}(x)$. [2]

3. EASY Calculator **[3 marks]**

The graph of $f(x) = -x^2 + 6x$ is shown below.



(a) Write down the coordinates of the vertex. [1]

(b) Write down the equation of the axis of symmetry. [1]

(c) State the range of f for $0 \leq x \leq 6$. [1]

4. EASY Calculator **[3 marks]**

The temperature, T degrees Celsius, inside a greenhouse t hours after midnight is modelled by

$$T(t) = 4\sin(15t)^\circ + 18, \qquad 0 \leq t \leq 24.$$

(a) Write down the amplitude of T. [1]

(b) Write down the period of T. [1]

(c) Write down the equation of the midline of T. [1]

5. **EASY** **Calculator** **[3 marks]**

The number of bacteria, N, in a culture after t hours is modelled by

$$N(t) = 500 \times 1.4^t, \qquad t \geq 0.$$

(a) Write down the initial number of bacteria. [1]

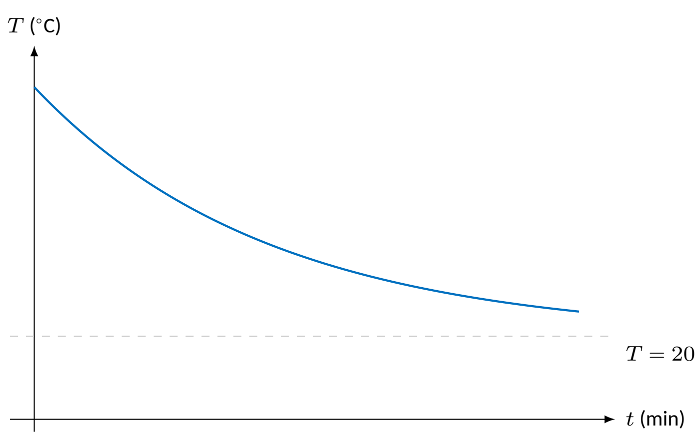
(b) Find the number of bacteria after 3 hours. Give your answer to the nearest integer. [2]

6. **EASY** **Calculator** **[3 marks]**

The cooling temperature, T degrees Celsius, of a cup of tea t minutes after it is made is modelled by

$$T(t) = 60 \times 0.95^t + 20, \qquad t \geq 0.$$

The graph of T is shown below.



(a) Write down the value of T when $t = 0$. [1]

(b) Write down the equation of the horizontal asymptote. [1]

(c) State what the horizontal asymptote represents in context. [1]

7. EASY Calculator **[3 marks]**

Two functions are defined as $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find the x-coordinates of the points where the graphs of f and g intersect. [2]

(b) Write down the y-coordinate of the intersection point that has the larger x-value. [1]

8. EASY Calculator **[3 marks]**

A cubic function is given by $h(x) = x^3 - 6x^2 + 9x$.

(a) Write down the x-intercepts of the graph of h. [2]

(b) Find the coordinates of the local maximum of h. [1]

9. EASY Calculator **[3 marks]**

A function is defined as $f(x) = \frac{1}{2}x + 5$ for $-4 \leq x \leq 8$.

(a) Write down the range of f. [2]

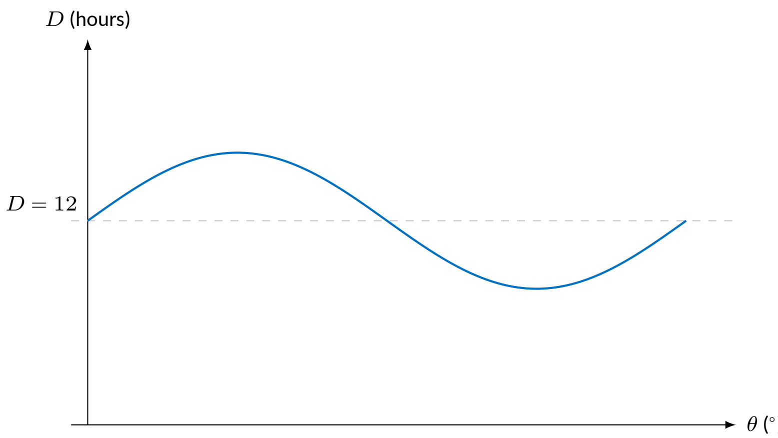
(b) Find $f^{-1}(8)$. [1]

10. **EASY** **Calculator** **[3 marks]**

The daily hours of daylight, D, in a city during a year are modelled by

$$D(\theta) = 3\sin(\theta)^\circ + 12, \qquad 0 \leq \theta \leq 360,$$

where θ represents the day of the year scaled to degrees. The graph of D is shown below.



(a) Write down the maximum number of hours of daylight. [1]

(b) Write down the minimum number of hours of daylight. [1]

(c) Find the value of θ for which $D(\theta) = 12$ for the first time after $\theta = 0$. [1]

Section B — Medium MEDIUM (10 questions $\times$ 4 marks = 40 marks)

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11. MEDIUM Calculator **[4 marks]**

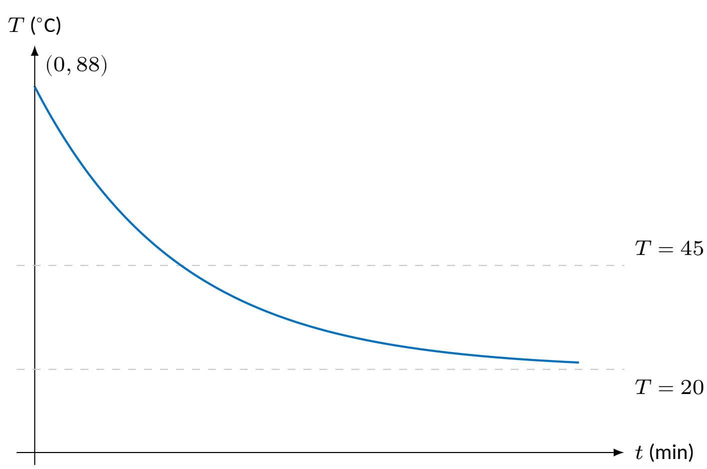
The temperature of a cup of tea, T (°C), at time t minutes after it is made, is modelled by

$$T(t) = 68 \cdot (0.94)^t + 20, \qquad t \geq 0.$$

(a) Write down the temperature of the tea when it is first made. [1]

(b) Find the value of t when $T = 45$. [2]

(c) State the equation of the horizontal asymptote of T and explain what it represents in context. [1]



12. **MEDIUM** **Calculator** **[4 marks]**

Let $f(x) = 2x^2 - 8x + 3$.

(a) Find the coordinates of the vertex of the graph of f. [2]

(b) Write down the equation of the axis of symmetry. [1]

(c) The graph of f passes through the point $(7, k)$. Find the value of k, and write down the other x-value that gives the same function value. [1]

13. **MEDIUM** **Calculator** **[4 marks]**

The function $g(x) = x^3 - 3x^2 - 9x + 5$ is defined for $-3 \leq x \leq 6$.

(a) Find the coordinates of the local maximum and local minimum of g. [2]

(b) Find the x-intercepts of g in this domain. [1]

(c) Write down the range of g on the given domain. [1]

14. MEDIUM Calculator **[4 marks]**

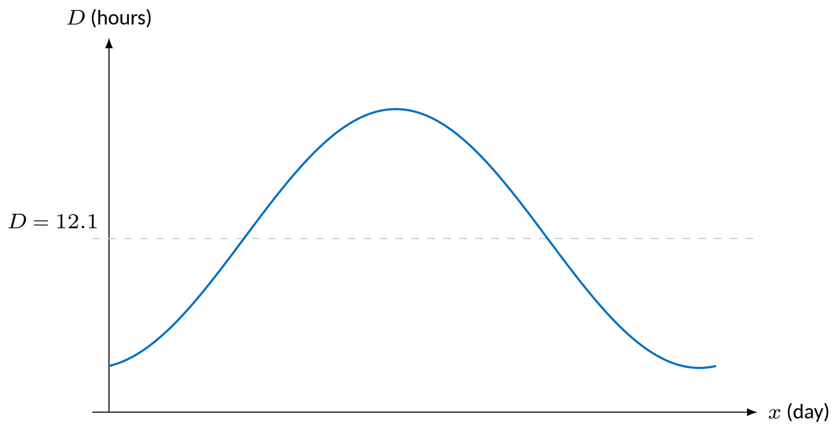
The number of hours of daylight in a city, D, on day x of the year is modelled by

$$D(x) = 3.8\sin(0.986x - 80)^\circ + 12.1, \qquad 0 \le x \le 365.$$

(a) Write down the amplitude and the midline of D. [2]

(b) Find the maximum number of hours of daylight predicted by the model. [1]

(c) Find the value of x, to the nearest day, on which D first equals 14 hours. [1]



15. MEDIUM Calculator **[4 marks]**

Let $f(x) = 3x + 7$.

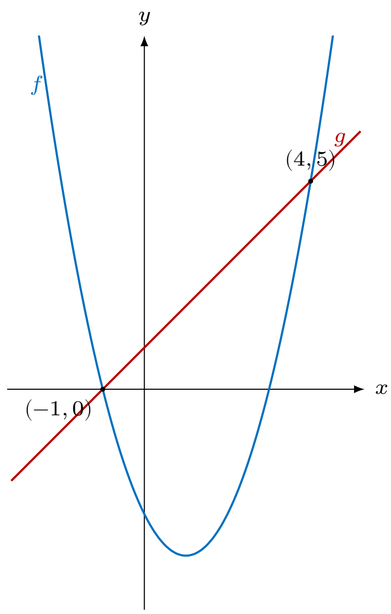
(a) Find $f^{-1}(x)$. [2]

(b) Evaluate $f^{-1}(22)$. [1]

(c) Explain what $f^{-1}(22)$ represents in context, given that $f(x)$ converts a temperature x (°C) into a value on a different scale. [1]

16. MEDIUM Calculator **[4 marks]**

The graphs of $f(x) = x^2 - 2x - 3$ and $g(x) = x + 1$ are shown below.



(a) Find the x-coordinates of the points where f and g intersect. [2]

(b) Write down the interval in which $g(x) > f(x)$. [1]

(c) Find the value of $f(x) - g(x)$ at $x = 3$. [1]

17. MEDIUM Calculator **[4 marks]**

A function is given by $h(x) = -2(x-1)^2 + 8$, for $-1 \leq x \leq 5$.

(a) Write down the coordinates of the vertex. [1]

(b) Find the x-intercepts of h. [2]

(c) State the range of h on its given domain. [1]

18. MEDIUM Calculator **[4 marks]**

A population of fish in a lake, P (hundreds), is modelled by

$$P(t) = 4(1.15)^t, \qquad t \geq 0,$$

where t is the time in years since monitoring began.

(a) Write down the initial population of fish. [1]

(b) Find the number of complete years it takes for the population to first exceed 1000 fish. [2]

(c) Determine whether the model predicts a population greater than 5000 fish after 25 years. Justify your answer. [1]

19. MEDIUM Calculator **[4 marks]**

Let $f(x) = \dfrac{12}{x-2} + 1$, for $x \neq 2$.

(a) Write down the equations of both asymptotes of f. [2]

(b) Find the x-intercept and the y-intercept of f. [2]

20. MEDIUM Calculator **[4 marks]**

The profit, P (thousands of dollars), of a small company after t years of operation is modelled by

$$P(t) = t^3 - 6t^2 + 9t - 1, \qquad 0 \leq t \leq 6.$$

(a) Find the local maximum value of P and the value of t at which it occurs. [2]

(b) Find the value of t at which P is first equal to 3 thousand dollars. [1]

(c) Write down the range of P on the domain $0 \leq t \leq 6$. [1]

Section C — Hard HARD (10 questions × 5 marks = 50 marks)

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21. HARD Calculator **[5 marks]**

The function $f(x) = x^3 - 3x^2 - 3x + k$ has a local maximum at $x = p$ and a local minimum at $x = q$, where $p < q$.

(a) Show that $p = 1 - \sqrt{2}$ and $q = 1 + \sqrt{2}$. [1]

(b) Find the value of k such that the local minimum value of f is zero. [3]

(c) Hence state the number of distinct positive real roots of $f(x) = 0$ when k takes the value found in part (b), justifying your answer. [1]

22. HARD Calculator **[5 marks]**

An open-top cylindrical container is to be made from 200 cm^2 of sheet material, using it all for the circular base and the curved lateral surface (no material for a lid). Let r be the base radius in centimetres and h the height in centimetres.

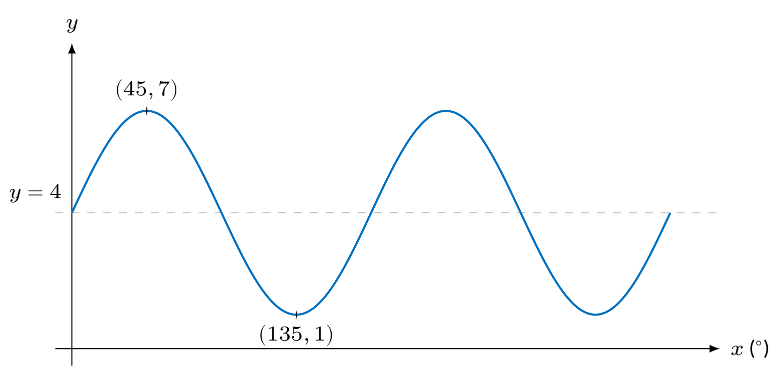
(a) Show that $h = \dfrac{100}{\pi r} - \dfrac{r}{2}$, and write down the exact domain of r for which $h > 0$. [1]

(b) Show that the volume of the container is $V(r) = 100r - \dfrac{\pi}{2} r^3$. [1]

(c) Find the value of r that gives the maximum volume and state this maximum volume in litres, correct to 3 significant figures. [3]

23. HARD Calculator **[5 marks]**

The graph of $y = f(x)$ is shown below, where $f(x) = a\sin(bx) + d$ for $0^\circ \leq x \leq 360^\circ$.



(a) Write down the values of a, b and d. [2]

(b) Find all values of x in the domain $0^\circ \leq x \leq 360^\circ$ for which $f(x) = 2.5$. [2]

(c) Determine whether $f(x) = 9$ has a solution in the given domain. Justify your answer. [1]

24. HARD Calculator **[5 marks]**

A scientist models the population of a colony of bacteria (in thousands) at time t hours using

$$P(t) = \frac{120}{1 + 5e^{-0.4t}}, \qquad t \geq 0.$$

(a) Write down the initial population of the colony. [1]

(b) Find the time at which the population first reaches 80 thousand. [2]

(c) State the equation of the horizontal asymptote of $P(t)$ and interpret its meaning in this context. [1]

(d) Determine whether the population ever reaches 130 thousand. Justify your answer. [1]

25. HARD Calculator **[5 marks]**

The functions f and g are defined for $x \in \mathbb{R}$ by

$$f(x) = 3x - 2, \qquad g(x) = x^2 + 1.$$

(a) Find the two values of x for which $f(x) = g(x)$, giving your answers correct to 3 significant figures. [2]

(b) Hence write down the interval where $g(x) > f(x)$. [1]

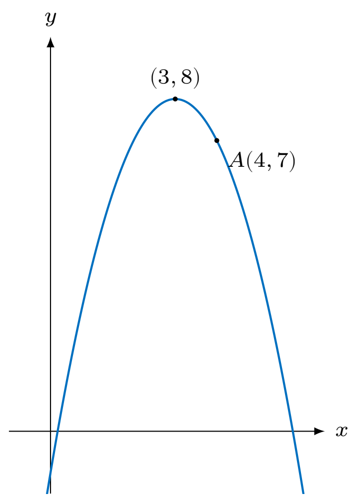
(c) The inverse function f^{-1} exists. Find $f^{-1}(x)$ and state what $f^{-1}(7)$ represents in context. [2]

26. **HARD** **Calculator** **[5 marks]**

The graph of $y = g(x)$ is shown below, where

$$g(x) = -x^2 + 6x + c.$$

The graph passes through the point $A(4, 7)$.



(a) Show that $c = -1$. [1]

(b) Write down the equation of the axis of symmetry and the coordinates of the vertex. [1]

(c) Find the x-intercepts of g, correct to 3 significant figures. [2]

(d) The second x-intercept can be found using the axis of symmetry. If one intercept is at $x = \alpha$, write down the other intercept in terms of α and the axis of symmetry. [1]

27. HARD Calculator **[5 marks]**

A function is defined by $f(x) = 2(3)^x - 18$ for $x \in \mathbb{R}$.

(a) Write down the equation of the horizontal asymptote of f. [1]

(b) Find the x- and y-intercepts of f. [2]

(c) The inverse function f^{-1} exists for the full domain of f.

(i) Find $f^{-1}(x)$. [1]

(ii) State the domain and range of f^{-1}. [1]

28. HARD Calculator **[5 marks]**

Two functions are given by

$$h(x) = x^3 - 6x^2 + 9x + 2 \qquad \text{and} \qquad k(x) = x + 2.$$

(a) Show that $h(x) = k(x)$ has exactly one solution at $x = 0$, and find the coordinates of the two other intersections. [3]

(b) Find the local maximum and local minimum of $h(x)$, giving coordinates correct to 3 significant figures. [1]

(c) Determine the interval(s) where $h(x) > k(x)$. Justify your answer with reference to your results from part (a). [1]

29. HARD Calculator **[5 marks]**

The number of hours of daylight in a city is modelled by

$$D(\theta) = 3.8\sin(\theta - 80) + 12, \qquad 0^\circ \leq \theta \leq 360^\circ,$$

where θ represents the day of the year scaled so that $\theta = 1^\circ$ corresponds to day 1.

(a) Write down the amplitude and the midline of D. [1]

(b) Find the two values of θ for which $D(\theta) = 14$ hours, correct to the nearest integer. [2]

(c) Find the maximum number of daylight hours and state the value of θ at which this occurs. [1]

(d) Determine whether $D(\theta) = 16.5$ is possible in this model. Justify your answer. [1]

30. HARD Calculator **[5 marks]**

The function $f(x) = \dfrac{x^2+4}{x-1}$ is defined for $x \in \mathbb{R}$, $x \neq 1$.

(a) Write down the equation of the vertical asymptote. [1]

(b) Find the coordinates of the local maximum and local minimum of f, correct to 3 significant figures. [2]

(c) Find the value(s) of x for which $f(x) = x + 1$. [1]

(d) Determine whether f has an oblique asymptote. Show that $f(x) = x + 1 + \dfrac{5}{x-1}$ and hence write down the equation of the oblique asymptote, justifying why f never equals this asymptote. [1]

Worked Solutions

Q1 — Function evaluation and solving EASY

(a) Substitute $x = 4$ into $f(x) = 3x - 7$: $\quad f(4) = 3(4) - 7 = 12 - 7 = 5$. **A1**

(b) Set $f(x) = 14$ and solve: $3x - 7 = 14 \implies 3x = 21 \implies x = 7$. **M1A1**

Warning: Do not confuse $f(4)$ with solving $f(x) = 4$; the number in the bracket is the input, not the output.

Q2 — Inverse function EASY

(a) $g(0) = 2(0)^2 - 8 = -8$. **A1**

(b) Write $y = 2x^2 - 8$, swap x and y, solve for y. Since the domain of g is $x \geq 0$, the inverse takes only the positive root:

$$x = 2y^2 - 8 \implies y^2 = \frac{x+8}{2} \implies g^{-1}(x) = \sqrt{\frac{x+8}{2}}, \quad x \geq -8.$$

M1A1

Warning: Because the domain of g is restricted to $x \geq 0$, only the positive square root is taken.

Q3 — Quadratic key features EASY

(a) Vertex $= (3, 9)$ (read from the graph). **A1**

(b) Axis of symmetry: $x = 3$. **A1**

(c) On $0 \leq x \leq 6$: $f(0) = 0$, rises to a maximum of 9, returns to $f(6) = 0$. Range: $0 \leq f(x) \leq 9$. **A1**

Warning: The range must be stated as an inequality in y-values, not x-values.

Q4 — Sinusoidal parameters EASY

(a) Amplitude $= 4$. **A1**

(b) Period $= \dfrac{360^\circ}{b}$ with $b = 15$: Period $= \dfrac{360}{15} = 24$ hours. **A1**

(c) Midline: $T = 18$. **A1**

Warning: The midline must be written as an equation, not just the number 18.

Q5 — Exponential model evaluation EASY

(a) $N(0) = 500 \times 1.4^0 = 500$. **A1**

(b) GDC: evaluate 500×1.4^3 on the home screen. $N(3) = 500 \times 2.744 = 1372$. **M1A1**

Warning: Do not round 1.4^3 prematurely before multiplying by 500.

Q6 — Exponential with asymptote EASY

(a) $T(0) = 60 \times 0.95^0 + 20 = 60 + 20 = 80$. **A1**

(b) As $t \to \infty$, $60 \times 0.95^t \to 0$, so $T \to 20$. Asymptote: $T = 20$. **A1**

(c) $T = 20$ represents the room (ambient) temperature that the tea approaches but never quite reaches. **A1**

Warning: The model never actually reaches $T = 20$; the tea approaches but does not attain this temperature.

Q7 — Intersecting graphs EASY

(a) GDC: Graph $Y_1 = 2x + 1$ and $Y_2 = x^2 - 3$; use Intersection. $2x + 1 = x^2 - 3 \implies x^2 - 2x - 4 = 0$. $x = -1.236\ldots$ or $x = 3.236\ldots$; to 3 s.f.: $x \approx -1.24$ and $x \approx 3.24$. **M1A1**

(b) $y = 2(3.236\ldots) + 1 = 7.472\ldots \approx 7.47$. **A1**

Warning: Use the unrounded x-value when computing y to avoid accumulating rounding error.

Q8 — Cubic function features EASY

(a) $h(x) = x^3 - 6x^2 + 9x = x(x^2 - 6x + 9) = x(x - 3)^2$. x-intercepts: $x = 0$ and $x = 3$. **A1A1**

(b) GDC: Graph $\to$ Maximum. Local maximum at $(1, 4)$. **A1**

Warning: $x = 3$ is a local minimum (touching the axis), not a maximum.

Q9 — Domain range and inverse EASY

(a) f is linear and increasing, so evaluate at both endpoints: $f(-4) = \frac{1}{2}(-4) + 5 = 3$; $f(8) = \frac{1}{2}(8) + 5 = 9$. Range: $3 \le f(x) \le 9$. **A1A1**

(b) Solve $f(x) = 8$: $\frac{1}{2}x + 5 = 8 \implies \frac{1}{2}x = 3 \implies x = 6$. **A1**

Warning: State the range using y-values, not x-values.

Q10 — Sinusoid reading EASY

(a) $D_{\text{max}} = 12 + 3 = 15$ hours. **A1**

(b) $D_{\text{min}} = 12 - 3 = 9$ hours. **A1**

(c) Solve $3\sin(\theta) + 12 = 12 \Rightarrow \sin(\theta) = 0$; the first solution after $\theta = 0$ is $\theta = 180^\circ$. **A1**
Warning: $\theta = 0$ also satisfies $\sin(\theta) = 0$ but is the starting point, not a crossing after $\theta = 0$.

Q11 — Exponential cooling model MEDIUM

(a) $T(0) = 68(0.94)^0 + 20 = 68 + 20 = 88\,^\circ$C. **A1**

(b) Solve $68(0.94)^t + 20 = 45$, i.e. $68(0.94)^t = 25$.
GDC: Graph $Y_1 = 68(0.94)^x + 20$ and $Y_2 = 45$; use Intersection.
Unrounded: $t = 16.1717\ldots$. **M1**
$t = 16.2$ minutes (3 s.f.). **A1**

(c) As $t \to \infty$, $(0.94)^t \to 0$, so $T \to 20$. Asymptote: $T = 20$, the ambient (room) temperature the tea approaches but never reaches. **A1**
Warning: Check by substitution: $T(16.172) = 45.00$ exactly. A common slip is mis-simplifying $\ln(25/68)/\ln(0.94)$; always verify the GDC's intersection value by direct substitution back into $T(t)$.

Q12 — Quadratic vertex and symmetry MEDIUM

(a) Axis: $x = -\dfrac{b}{2a} = -\dfrac{-8}{2(2)} = 2$. $f(2) = 2(4) - 8(2) + 3 = 8 - 16 + 3 = -5$. Vertex: $(2, -5)$. **M1A1**

(b) Axis of symmetry: $x = 2$. **A1**

(c) $f(7) = 2(49) - 8(7) + 3 = 98 - 56 + 3 = 45$, so $k = 45$. Symmetry partner of $x = 7$ about $x = 2$: $x = 2 - 5 = -3$. **A1**
Warning: Use the axis of symmetry to find the partner point rather than guessing.

Q13 — Cubic key features MEDIUM

(a) GDC: Graph $y = x^3 - 3x^2 - 9x + 5$ on $[-3, 6]$; use Maximum and Minimum.
Local maximum: $(-1, 10)$. Local minimum: $(3, -22)$. **A1A1**

(b) GDC: use Zero (root finder). x-intercepts at $x \approx -2.18, 0.489, 4.69$ (3 s.f.). **A1**

(c) Endpoint values: $g(-3) = -22$; $g(6) = 216 - 108 - 54 + 5 = 59$. Global minimum -22 (at $x = 3$), global maximum 59 (at $x = 6$). Range: $-22 \le g(x) \le 59$. **A1**
Warning: Check endpoint values as well as local extrema — verified by the cubic root finder (numpy.roots): $-2.180, 0.489, 4.691$.

Q14 — Sinusoidal daylight model MEDIUM

(a) Amplitude $|a| = 3.8$ hours. Midline: $D = 12.1$. **A1A1**

(b) Maximum = midline + amplitude $= 12.1 + 3.8 = 15.9$ hours. **A1**

(c) Solve $3.8\sin(0.986x - 80)^\circ + 12.1 = 14$.
GDC: Graph $Y_1 = 3.8\sin(0.986x - 80) + 12.1$ (degree mode) and $Y_2 = 14$; first intersection on $[0, 365]$.
Unrounded: $x = 111.56\ldots$ days. **M1**
$x \approx 112$ (nearest day). **A1**
Warning: Ensure the GDC is in degree mode. The argument $(0.986x - 80)$ already has degrees built in, so no unit conversion is needed. The first crossing occurs while D is still rising toward its maximum at $x \approx 172$, well past day 100.

Q15 — Inverse function in context MEDIUM

(a) Set $y = 3x + 7$, swap x and y: $x = 3y + 7 \Rightarrow y = \frac{x-7}{3}$. $f^{-1}(x) = \frac{x-7}{3}$. **M1A1**

(b) $f^{-1}(22) = \frac{22-7}{3} = \frac{15}{3} = 5$. **A1**

(c) $f^{-1}(22) = 5$ means a reading of 22 on the new scale corresponds to $5\,^\circ$C on the original scale. **A1**
Warning: An inverse function is not the reciprocal; always derive f^{-1} by solving.

Q16 — Intersecting graphs MEDIUM

(a) Set $f(x) = g(x)$: $x^2 - 2x - 3 = x + 1 \Rightarrow x^2 - 3x - 4 = 0$. GDC: Intersection, or factorise: $x = -1$ and $x = 4$. **M1A1**

(b) Between the two intersections, $g(x) > f(x)$: $-1 < x < 4$. **A1**

(c) $f(3) - g(3) = (9 - 6 - 3) - (3 + 1) = 0 - 4 = -4$. **A1**
Warning: The question asks for $f(x) - g(x)$, not $g(x) - f(x)$; a sign error here costs the mark.

Q17 — Vertex form quadratic on restricted domain MEDIUM

(a) Vertex form $-2(x-1)^2 + 8$; vertex is $(1, 8)$. **A1**

(b) $-2(x-1)^2 + 8 = 0 \Rightarrow (x-1)^2 = 4 \Rightarrow x - 1 = \pm 2$. $x = 3$ or $x = -1$. **M1A1**

(c) $h(-1) = -2(4)+8 = 0$; $h(5) = -2(16)+8 = -24$; maximum at vertex is 8. Range: $-24 \le h(x) \le 8$. **A1**
Warning: The domain is $[-1, 5]$ — always evaluate at both endpoints and the vertex on a closed domain.

Q18 — Exponential population growth MEDIUM

(a) $P(0) = 4(1.15)^0 = 4$ hundreds $= 400$ fish. **A1**

(b) Need $P(t) > 10$ (since 1000 fish $= 10$ hundreds): $4(1.15)^t > 10$.
GDC: Graph $Y_1 = 4(1.15)^x$ and $Y_2 = 10$; use Intersection. Unrounded crossing: $t = 6.556\ldots$ years.
Check integer years: $P(6) = 4(1.15)^6 = 9.25\ldots < 10$; $P(7) = 4(1.15)^7 = 10.64\ldots > 10$. **M1**
First complete year: $t = 7$ years. **A1**

(c) $P(25) = 4(1.15)^{25} = 4(32.918\ldots) = 131.67\ldots$ hundreds $= 13\,167$ fish. Since $13\,167 > 5000$, yes, the model predicts a population greater than 5000 fish after 25 years. **A1**

Warning: P is in hundreds — convert carefully. For part (b), always bracket with the integer years either side of the unrounded crossing.

Q19 — Rational function asymptotes and intercepts MEDIUM

(a) Vertical asymptote: denominator zero when $x = 2$; equation: $x = 2$.
Horizontal asymptote: as $x \to \pm\infty$, $\dfrac{12}{x-2} \to 0$, so $f(x) \to 1$; equation: $y = 1$. **A1A1**

(b) x-intercept: $\dfrac{12}{x-2} + 1 = 0 \Rightarrow x - 2 = -12 \Rightarrow x = -10$. **M1**
y-intercept: $f(0) = \dfrac{12}{0-2} + 1 = -6 + 1 = -5$. **A1**
Warning: The horizontal asymptote is $y = 1$, not $y = 0$ — the $+1$ shifts the whole graph up.

Q20 — Cubic profit model MEDIUM

(a) GDC: Graph $y = t^3 - 6t^2 + 9t - 1$ on $[0, 6]$; use Maximum. Local maximum at $t = 1$, $P = 3$. **M1A1**

(b) P first equals 3 at $t = 1$ year (the local maximum). **A1**

(c) Endpoint values: $P(0) = -1$; $P(6) = 216 - 216 + 54 - 1 = 53$. Local minimum (GDC): at $t = 3$, $P = 27 - 54 + 27 - 1 = -1$. Range: $-1 \le P(t) \le 53$. **A1**
Warning: The global maximum occurs at the right endpoint $t = 6$, not at the local maximum $t = 1$.

Q21 — Cubic local extrema and root count HARD

(a) Differentiate $f(x) = x^3 - 3x^2 - 3x + k$: $f'(x) = 3x^2 - 6x - 3$. Setting $f'(x) = 0$:

$$x = \frac{6 \pm \sqrt{36+36}}{6} = \frac{6 \pm \sqrt{72}}{6} = \frac{6 \pm 6\sqrt{2}}{6} = 1 \pm \sqrt{2}.$$

So $p = 1 - \sqrt{2}$ (local max, since $f''(p) = 6x - 6 < 0$ here) and $q = 1 + \sqrt{2}$ (local min, since $f''(q) > 0$). **AG**

(b) We require $f(q) = 0$ where $q = 1 + \sqrt{2}$. **M1**
GDC / exact algebra: $f(q) = q^3 - 3q^2 - 3q + k = 0 \implies k = -q^3 + 3q^2 + 3q$.
Using $q = 1 + \sqrt{2} \approx 2.41421$: $k = 5 + 4\sqrt{2} = 10.6569\ldots$ **M1**
$k = 10.7$ (3 s.f.). **A1**

(c) With this k, the cubic touches zero at $x = q \approx 2.41$ (a repeated root, since the local minimum value is exactly zero there) and crosses zero once more at $x = 1 - 2\sqrt{2} \approx -1.83$, which is negative. **R1**
There is exactly **one** distinct positive real root ($x = q$, with multiplicity two); the only other root is negative.
Warning: A repeated root counts as one distinct root value, not two, even though it has multiplicity two. Always check the sign of every root, not just its existence.

Q22 — Cylinder volume optimization HARD

(a) Total material: base + lateral surface $= \pi r^2 + 2\pi rh = 200$.
Solve for h: $2\pi rh = 200 - \pi r^2 \implies h = \dfrac{200 - \pi r^2}{2\pi r} = \dfrac{100}{\pi r} - \dfrac{r}{2}$. **M1**
Domain requires $h > 0$: $\dfrac{100}{\pi r} > \dfrac{r}{2} \implies r^2 < \dfrac{200}{\pi} \implies r < \sqrt{\dfrac{200}{\pi}}$. Also $r > 0$.
Domain: $0 < r < \sqrt{200/\pi} \approx 7.98$ cm. **A1**

(b) $V = \pi r^2 h = \pi r^2 \left(\dfrac{100}{\pi r} - \dfrac{r}{2}\right) = 100r - \dfrac{\pi}{2}r^3$. **AG**

(c) $\dfrac{dV}{dr} = 100 - \dfrac{3\pi}{2}r^2$. Setting $\dfrac{dV}{dr} = 0$: **M1**

$$r^2 = \frac{200}{3\pi} \implies r = \sqrt{\frac{200}{3\pi}} = 4.6066\ldots$$

$\dfrac{d^2V}{dr^2} = -3\pi r < 0$ at this r, confirming a maximum. **A1**
$r = 4.61$ cm (3 s.f.).
$V_{\text{max}} = 100(4.6066\ldots) - \dfrac{\pi}{2}(4.6066\ldots)^3 = 307.106\ldots$ cm^3.
Converting to litres: $307.106/1000 = 0.307106\ldots$ litres.
$V_{\text{max}} \approx 0.307$ litres (3 s.f.). **A1**
Warning: Convert cm^3 to litres by dividing by 1000. Always confirm a stationary point is a maximum (not just find where $dV/dr = 0$) using the second derivative.

Q23 — Sinusoid parameter identification and solve HARD

(a) Maximum 7, minimum 1: amplitude $a = \dfrac{7-1}{2} = 3$; midline $d = \dfrac{7+1}{2} = 4$. Maximum at $x = 45^\circ$, minimum at $x = 135^\circ$, giving a half-period of 90°, so period $= 180^\circ$. Since period $= \dfrac{360^\circ}{b}$, $b = 2$. **A1A1**

$$a = 3, \quad b = 2, \quad d = 4.$$

(b) Solve $3\sin(2x) + 4 = 2.5 \Rightarrow \sin(2x) = -0.5$ for $0^\circ \le x \le 360^\circ$. **M1**
$2x = 210^\circ, 330^\circ, 570^\circ, 690^\circ \implies x = 105^\circ, 165^\circ, 285^\circ, 345^\circ$. **A1**

(c) Maximum value of f is $d + a = 4 + 3 = 7$. Since $9 > 7$, $f(x) = 9$ has no solution. **R1**

Warning: The period here is 180°, not 360° — there are two full cycles in the given domain, giving four solutions in part (b).

Q24 — Logistic-style exponential asymptote and bounds HARD

(a) $P(0) = \dfrac{120}{1+5e^0} = \dfrac{120}{6} = 20$ thousand. **A1**

(b) Solve $\dfrac{120}{1+5e^{-0.4t}} = 80$: **M1**

$$1 + 5e^{-0.4t} = 1.5 \implies 5e^{-0.4t} = 0.5 \implies e^{-0.4t} = 0.1.$$

$-0.4t = \ln(0.1) \implies t = \dfrac{-\ln(0.1)}{0.4} = \dfrac{\ln 10}{0.4} = 5.7565\ldots$
$t \approx 5.76$ hours (3 s.f.). **A1**

(c) As $t \to \infty$, $e^{-0.4t} \to 0$, so $P(t) \to \dfrac{120}{1} = 120$. Asymptote: $P = 120$; the population approaches but never exceeds 120 thousand in the long run. **A1**

(d) Since $P(t) < 120$ for all finite t (strictly increasing and bounded above by 120), $P(t) = 130 > 120$ is impossible. **R1**

Warning: The asymptote is a strict upper bound that is never attained, however large t becomes.

Q25 — Intersecting graphs interval comparison and inverse HARD

(a) Set $f(x) = g(x)$: $3x - 2 = x^2 + 1 \Rightarrow x^2 - 3x + 3 = 0$. **M1**
Discriminant $= 9 - 12 = -3 < 0$: no real solutions — f and g do not intersect. **A1**

(b) Since the graphs never intersect and $g(0) = 1 > -2 = f(0)$, $g(x) > f(x)$ for all $x \in \mathbb{R}$. **A1**

(c) Let $y = 3x - 2$, swap and solve: $x = 3y - 2 \Rightarrow y = \dfrac{x+2}{3}$. **M1**

$$f^{-1}(x) = \frac{x+2}{3}.$$

$f^{-1}(7) = \dfrac{9}{3} = 3$. In context: $f^{-1}(7)$ gives the input $x = 3$ that maps to output 7 under f. **A1**

Warning: Always verify the discriminant before attempting a numerical solve that will not converge.

Q26 — Quadratic show constant and symmetry partner HARD

(a) Substitute $A(4, 7)$ into $g(x) = -x^2 + 6x + c$: **M1**

$$7 = -(4)^2 + 6(4) + c = -16 + 24 + c = 8 + c \implies c = -1.$$

AG

(b) Axis of symmetry: $x = -\dfrac{b}{2a} = -\dfrac{6}{2(-1)} = 3$. Vertex: $g(3) = -9 + 18 - 1 = 8$, so vertex $= (3, 8)$. **A1**

(c) Solve $-x^2 + 6x - 1 = 0 \Rightarrow x^2 - 6x + 1 = 0 \Rightarrow x = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$. **M1**
$x \approx 0.172$ or $x \approx 5.83$ (3 s.f.). **A1**

(d) By symmetry about $x = 3$: if one intercept is at $x = \alpha$, the other is at $x = 6 - \alpha$ (reflecting α in the axis $x = 3$). **A1**

Warning: Always verify a "show that" target by direct substitution before trusting it — here $g(4) = -16 + 24 - 1 = 7$✓, confirming $c = -1$ is consistent with both the given point and the vertex shown on the graph.

Q27 — Exponential intercepts and inverse domain/range HARD

(a) As $x \to -\infty$, $2(3)^x \to 0$, so $f(x) \to -18$. Horizontal asymptote: $y = -18$. **A1**

(b) y-intercept: $f(0) = 2(3)^0 - 18 = 2 - 18 = -16$, so $(0, -16)$.
x-intercept: $2(3)^x = 18 \Rightarrow (3)^x = 9 = 3^2 \Rightarrow x = 2$, so $(2, 0)$. **A1A1**

(c)(i) Let $y = 2(3)^x - 18$. Swap and solve: **M1**

$$x = 2(3)^y - 18 \implies (3)^y = \frac{x+18}{2} \implies y = \log_3\left(\frac{x+18}{2}\right) = \frac{\ln\left(\frac{x+18}{2}\right)}{\ln 3}.$$

$$f^{-1}(x) = \frac{\ln\left(\frac{x+18}{2}\right)}{\ln 3}.$$

A1

(c)(ii) The domain of f^{-1} equals the range of f. Since f has range $(-18, +\infty)$, the domain of f^{-1} is

$x > -18$. The range of f^{-1} equals the domain of f: $y \in \mathbb{R}$. **A1**

Warning: The exponential never attains its asymptote, so -18 is excluded from the domain of f^{-1} — use a strict inequality.

Q28 — Cubic meets line extrema and interval inequality HARD

(a) Set $h(x) = k(x)$: $x^3 - 6x^2 + 9x + 2 = x + 2 \Rightarrow x^3 - 6x^2 + 8x = 0 \Rightarrow x(x^2 - 6x + 8) = 0$. **M1**
Factorising: $x(x-2)(x-4) = 0$, giving $x = 0, 2, 4$. This confirms $x = 0$ is a solution. **A1**
Other intersections: $x = 2$, $k(2) = 4 \Rightarrow (2, 4)$; $x = 4$, $k(4) = 6 \Rightarrow (4, 6)$. **A1**

(b) GDC: Graph $y = x^3 - 6x^2 + 9x + 2$, use Maximum/Minimum.
Local maximum $(1, 6.00)$; local minimum $(3, 2.00)$ (exact: $h(1) = 6$, $h(3) = 2$). **A2**

(c) Test each interval: at $x = 1$: $h = 6$, $k = 3$, so $h > k$. At $x = 3$: $h = 2$, $k = 5$, so $h < k$. At $x = 5$: $h = 22$, $k = 7$, so $h > k$. At $x = -1$: $h = -14$, $k = 1$, so $h < k$. **R1**
Therefore $h(x) > k(x)$ on $(0, 2) \cup (4, +\infty)$. **A1**

Warning: The inequality is strict, so endpoints where $h = k$ are excluded. Don't forget to check $x < 0$ as well.

Q29 — Sinusoid daylight hours solve and bounds HARD

(a) Amplitude $|a| = 3.8$ hours. Midline: $D = 12$ hours. **A1**

(b) Solve $3.8\sin(\theta - 80) + 12 = 14 \Rightarrow \sin(\theta - 80) = \dfrac{2}{3.8} = 0.52631\ldots$ **M1**
Let $\alpha = \theta - 80$: $\alpha_1 = \arcsin(0.52631\ldots) \approx 31.74^\circ$, so $\theta_1 = 80 + 31.74 \approx 112^\circ$.
$\alpha_2 = 180 - 31.74 = 148.26^\circ$, so $\theta_2 = 80 + 148.26 = 228.26 \approx 228^\circ$. **A1**

(c) Maximum occurs when $\sin(\theta - 80) = 1$: $\theta - 80 = 90^\circ \Rightarrow \theta = 170^\circ$. Maximum daylight $= 3.8(1) + 12 = 15.8$ hours. **A1**

(d) Maximum value is 15.8 hours. Since $16.5 > 15.8$, $D(\theta) = 16.5$ is impossible. **R1**

Warning: There are two solutions in $[0^\circ, 360^\circ]$ in part (b), because sine is positive in both Q1 and Q2 — only finding the first loses a mark.

Q30 — Rational function asymptotes and local extrema HARD

(a) Vertical asymptote: denominator zero when $x - 1 = 0 \Rightarrow x = 1$. **A1**

(b) $f'(x) = \dfrac{x^2 - 2x - 4}{(x-1)^2}$. Setting the numerator to zero: $x = \dfrac{2 \pm \sqrt{4 + 16}}{2} = 1 \pm \sqrt{5}$. **M1**
Local maximum at $x = 1 - \sqrt{5} \approx -1.236$: $f \approx -2.47$.
Local minimum at $x = 1 + \sqrt{5} \approx 3.236$: $f \approx 6.47$.
Local maximum: $(-1.24, -2.47)$; local minimum: $(3.24, 6.47)$ (3 s.f.). **A1**

(c) Set $\dfrac{x^2+4}{x-1} = x+1$: $x^2+4 = (x+1)(x-1) = x^2-1 \Rightarrow 4 = -1$, a contradiction. **A1**
There are no values of x for which $f(x) = x+1$.

(d) Polynomial division: $x^2+4 = (x-1)(x+1)+5$, so **(M1)**

$$f(x) = \frac{(x-1)(x+1)+5}{x-1} = x+1+\frac{5}{x-1}.$$

AG

As $x \to \pm\infty$, $\dfrac{5}{x-1} \to 0$, so $f(x) \to x+1$. Oblique asymptote: $y = x+1$.

Since $f(x) = x+1$ requires $\dfrac{5}{x-1} = 0$, which has no solution (part (c)), f never attains $y = x+1$. **A1**

Warning: Don't forget the remainder when performing polynomial division — omitting it gives the wrong (unshifted) asymptote.