IB Math AI SL — Topic 3

Voronoi Diagrams

Drawing Voronoi Diagrams · Interpreting Cells · Largest Empty Circle

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. A graphic display calculator is required throughout; support GDC answers with the values entered and the feature used.

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Key Formulae

- **Distance between two points:** $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- **Midpoint of a segment:** $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$
- **Gradient of a segment:** $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
- **Perpendicular bisector gradient:** $m_\perp = -\dfrac{1}{m}$ (product of perpendicular gradients $= -1$)
- **Equation of a straight line (point–slope):** $y - y_1 = m(x - x_1)$
- **Voronoi vertex equidistance property:** At a Voronoi vertex V, $d(V, A) = d(V, B) = d(V, C)$ for all adjacent sites A, B, C
- **Largest empty circle (toxic waste dump):** The optimal location is the Voronoi vertex V^* that *maximises* $d(V, \text{nearest site})$; i.e. $V^* = \arg\max_V \min_i \, d(V, S_i)$

GDC Tips

- **Distance between two coordinate points:** Enter $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ directly into the Home screen; store intermediate values to avoid rounding errors across chained parts.
- **Finding the perpendicular bisector:** Use the GDC's `Numeric Solve` (or `solve()` on CAS) to verify the intersection of two bisector equations — input both equations simultaneously to find the Voronoi vertex coordinates.
- **Checking equidistance at a vertex:** Store the vertex coordinates, then evaluate the distance formula to each adjacent site on the Home screen; all three distances must match to at least 3 significant figures before proceeding.
- **Comparing candidate vertex distances (toxic waste dump):** Build a table of $d(V_i, \text{nearest site})$ for every labelled vertex using repeated distance calculations; use `Lists & Spreadsheet` or a table on the Home screen so all values are visible together for the final comparison.
- **Area of a polygonal cell:** If the cell is irregular, split it into triangles using $\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$ entered directly on the Home screen for each triangle, then sum.

Common Mistakes to Avoid

1. **Single-distance shortcut in nearest-site problems.** Calculating the distance from a query point to only one candidate site and concluding it belongs to that cell is *insufficient*. You must compute distances to **all** plausible neighbouring sites and compare; the examiner requires evidence that all alternatives are farther away.

2. **Using the wrong gradient for the perpendicular bisector.** A common error is using the gradient of the original segment instead of its negative reciprocal. Always explicitly write $m_\perp = -1/m$ and check the sign; this step is almost always assessed in "show that" parts and an incorrect gradient earns no method marks.

3. **Forgetting to verify equidistance at the Voronoi vertex.** Candidates find the intersection of two bisectors and assume correctness without checking that the vertex is also equidistant from the third adjacent site. Always confirm $d(V, A) = d(V, B) = d(V, C)$ numerically; a mismatch signals an algebra or rounding error earlier in the chain.

4. **Minimising instead of maximising in the toxic waste dump problem.** The largest empty circle requires the vertex that **maximises** the distance to the nearest site. Students frequently choose the vertex closest to all sites (the minimum), which is the *worst* location for a dump. State explicitly: "choose the vertex with the greatest nearest-site distance" in your justification (R1).

5. **Examining only some Voronoi vertices in the toxic waste dump problem.** Every labelled vertex is a candidate location; omitting even one can lead to the wrong answer. Tabulate all vertex distances and only then select the maximum. A partial table with a correct answer earns no reasoning mark if an unlisted vertex could have been larger.

6. **Premature rounding causing bisector intersection errors.** Rounding the gradient or midpoint to 3 s.f. before forming the bisector equation introduces error that compounds when the two bisectors are solved simultaneously. Keep full calculator precision (or exact fractions) throughout the chain and only round the *final* answer; always show the unrounded value before stating the rounded result.

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Section A — Easy

EASY

1. Two food trucks, $A(2, 4)$ and $B(8, 4)$, operate in a city modelled on a coordinate grid where each unit represents 1 km. The perpendicular bisector of AB forms the boundary between their delivery zones.

(a) Write down the coordinates of the midpoint M of AB. **[1 marks]**

(b) Write down the equation of the perpendicular bisector of AB. **[2 marks]**

2. Three parks are located at $P(1, 5)$, $Q(7, 5)$ and $R(4, 1)$ on a coordinate grid where each unit represents 1 km. A Voronoi diagram is constructed for the three sites.
The Voronoi vertex V lies at $(4, 3)$.
A new café is to be built at the point $C(3, 5)$. Determine which park is nearest to the café by calculating the distance from C to each of P and Q. **[3 marks]**

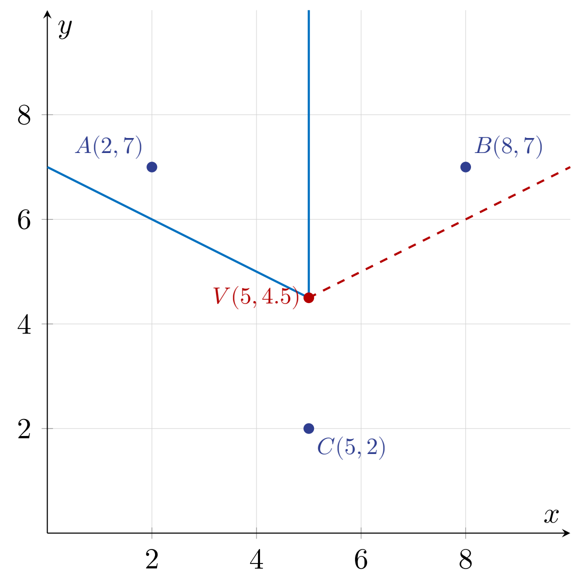
3. Two recycling centres are located at $A(1,3)$ and $B(5,7)$ on a coordinate grid where each unit represents 1 km.

(a) Find the gradient of AB. **[1 marks]**

(b) Find the gradient of the perpendicular bisector of AB. **[1 marks]**

(c) Write down the coordinates of the midpoint of AB. **[1 marks]**

4. The diagram below shows three mobile phone towers at $A(2,7)$, $B(8,7)$ and $C(5,2)$ on a coordinate grid where each unit represents 1 km. Two Voronoi edges are already drawn. The dashed line shows the missing third edge.
Identify the two sites that the dashed edge bisects and write down the equation of that edge.

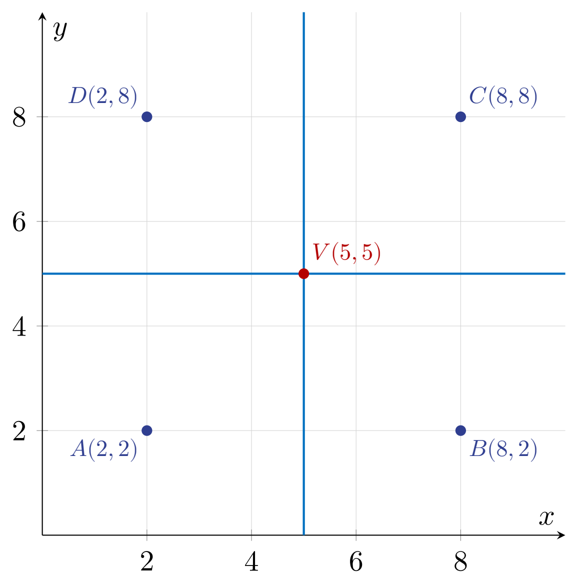


[3 marks]

5. A Voronoi diagram is used to model service zones for four hospitals located at $A(1,1)$, $B(9,1)$, $C(9,9)$ and $D(1,9)$ on a coordinate grid where each unit represents 1 km.
Write down the coordinates of the Voronoi vertex shared by all four cells. **[1 marks]**
Write down the equations of all four Voronoi edges. **[2 marks]**

6. Two surf schools operate at $S(2,3)$ and $T(8,3)$ on a coastal coordinate grid where each unit represents 1 km. A point $W(5,7)$ lies inside the region and a customer wishes to know which surf school is closer.
Calculate the distance from W to each of S and T, and state which surf school the customer should attend. **[3 marks]**

7. The diagram below shows four sites on a Voronoi diagram representing delivery depots at $A(2,2)$, $B(8,2)$, $C(8,8)$ and $D(2,8)$ on a coordinate grid where each unit represents 1 km. The Voronoi vertex V is marked.
A storage facility is to be placed at the Voronoi vertex $V(5,5)$, chosen because it is the point furthest from all four depots (the toxic waste dump problem).
Calculate the distance from V to site A and confirm that V is equidistant from all four sites.



[3 marks]

8. Two community centres are located at $P(0,4)$ and $Q(6,4)$ on a coordinate grid where each unit represents 1 km.

(a) Write down the midpoint M of PQ. **[1 marks]**

(b) A resident lives at point $R(3,1)$. Determine whether R lies in the Voronoi cell of P or Q by calculating the distance from R to each site. **[2 marks]**

9. Two fire stations are placed at $F(1,2)$ and $G(7,6)$ on a coordinate grid where each unit represents 1 km.

(a) Find the coordinates of the midpoint M of FG. **[1 marks]**

(b) Find the gradient of FG. **[1 marks]**

(c) Hence write down the gradient of the perpendicular bisector of FG. **[1 marks]**

10. Three weather stations are located at $A(1,6)$, $B(7,6)$ and $C(4,2)$ on a coordinate grid where each unit represents 1 km. The Voronoi vertex for these three sites is at $V(4,5)$.
A researcher records data at the point $D(6,4)$. By calculating the distance from D to each of A, B and C, determine which weather station's Voronoi cell contains D. **[3 marks]**

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Section B — Medium

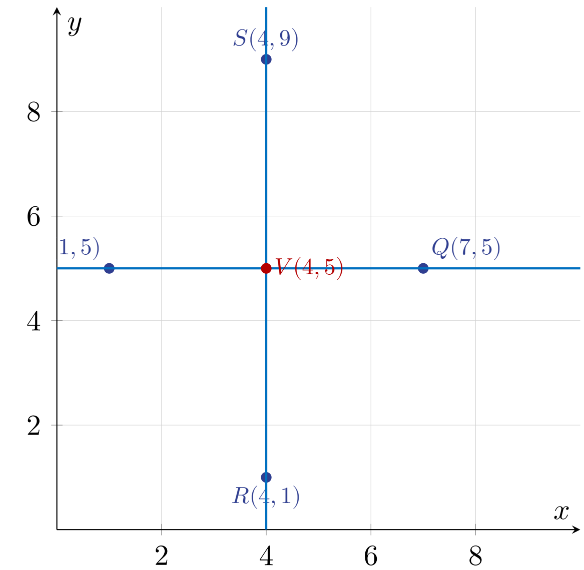
MEDIUM

11. Three food trucks operate at fixed positions in a city park. Their coordinates, in metres, are $A(2, 3)$, $B(8, 3)$, and $C(5, 9)$. A Voronoi diagram is being constructed to show which truck is closest to any point in the park.

(a) Find the equation of the perpendicular bisector of segment AB. **[2 marks]**

(b) A customer is standing at point $P(4, 7)$. Determine which food truck is nearest to the customer. Justify your answer. **[2 marks]**

12. Four wifi hotspots are installed in a shopping centre. Their coordinates, in metres, are $P(1, 5)$, $Q(7, 5)$, $R(4, 1)$, and $S(4, 9)$. Part of the Voronoi diagram is shown below, with Voronoi vertex V already marked.



(a) Write down the coordinates of the Voronoi vertex V. **[1 marks]**

(b) Verify that V is equidistant from sites P, Q, R, and S by calculating the distance from V to each site. **[3 marks]**

13. Three recycling depots are located in a region. Their coordinates, in kilometres, are $D_1(1,2)$, $D_2(7,2)$, and $D_3(4,8)$. A toxic waste container must be placed as far as possible from all three depots, at the Voronoi vertex of the diagram.

(a) Show that the perpendicular bisector of D_1D_2 has equation $x = 4$. **[2 marks]**

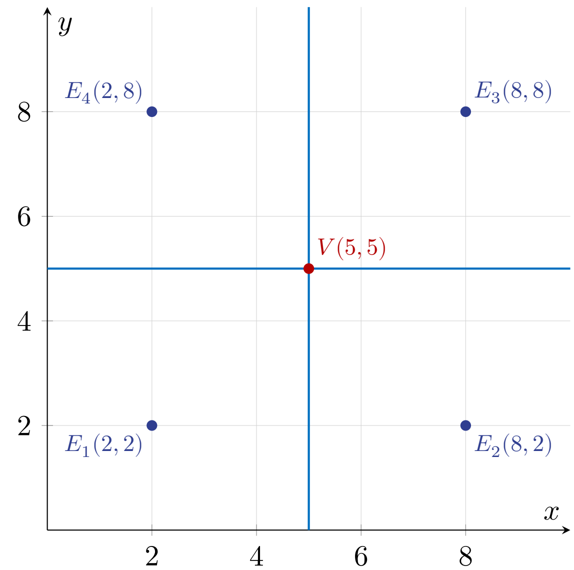
(b) The Voronoi vertex is at $V(4,k)$. Given that V is equidistant from D_1 and D_3, find the value of k. **[2 marks]**

14. Three weather monitoring stations are positioned at $A(0,4)$, $B(6,4)$, and $C(3,10)$ on a coordinate grid where units are in kilometres. The Voronoi diagram for these three stations has a single interior vertex V.

(a) Find the midpoint M of segment AC. **[1 marks]**

(b) Find the equation of the perpendicular bisector of AC. **[3 marks]**

15. A Voronoi diagram has been constructed for four emergency service stations at $E_1(2,2)$, $E_2(8,2)$, $E_3(8,8)$, and $E_4(2,8)$ on a coordinate grid where units are in kilometres. The diagram is shown below.



An incident occurs at point $T(3,7)$.

(a) Determine which station is nearest to T. Justify your answer by calculating distances to all plausible stations. **[3 marks]**

(b) State the equation of the Voronoi edge that separates the cells of E_3 and E_4. **[1 marks]**

16. Three post offices are located at $F(1,1)$, $G(9,3)$, and $H(5,9)$ on a map where units are in kilometres. A Voronoi diagram is to be constructed.

(a) Find the gradient of segment FG. **[1 marks]**

(b) Find the equation of the perpendicular bisector of FG. **[3 marks]**

17. Four campsites are located at $W(1,1)$, $X(7,1)$, $Y(7,7)$, and $Z(1,7)$ on a coordinate grid where units are in kilometres. A new campsite $N(4,4)$ is added to the region.

(a) State which existing campsite cells will be reduced in size when N is added. **[1 marks]**

(b) Find the equation of the perpendicular bisector of the segment joining $N(4,4)$ and $X(7,1)$. **[3 marks]**

18. A Voronoi diagram is constructed for three sites: $J(1, 6)$, $K(5, 2)$, and $L(9, 6)$ on a coordinate grid where units are in metres. The perpendicular bisectors of JK and KL intersect at Voronoi vertex V.

(a) Show that the perpendicular bisector of JK has equation $y = x + 1$. **[2 marks]**

(b) The perpendicular bisector of KL has equation $y = -x + 9$. Find the coordinates of vertex V. **[2 marks]**

19. Three toxic waste sites are located at $A(2, 8)$, $B(8, 8)$, and $C(5, 2)$ on a coordinate grid where units are in kilometres. A regulation requires that a new research facility be placed at the location that maximises the minimum distance to any waste site. The candidate locations are the Voronoi vertices of the diagram.
The three Voronoi vertices have been found to be $V_1(5, 8)$, $V_2(2, 5)$, and $V_3(8, 5)$.

(a) Calculate the distance from V_1 to its nearest waste site. **[1 marks]**

(b) Calculate the distance from V_2 to its nearest waste site and from V_3 to its nearest waste site. **[2 marks]**

(c) Hence determine at which vertex the research facility should be placed. Justify your answer. **[1 marks]**

20. Three internet relay towers are located at $P(0,0)$, $Q(8,0)$, and $R(4,6)$ on a coordinate grid where units are in kilometres. The Voronoi diagram for the three towers has a single interior vertex V.

(a) Write down the equation of the perpendicular bisector of PQ. **[1 marks]**

(b) Show that the perpendicular bisector of PR has equation $y = -\frac{2}{3}x + \frac{13}{3}$. **[2 marks]**

(c) Hence find the coordinates of vertex V. **[1 marks]**

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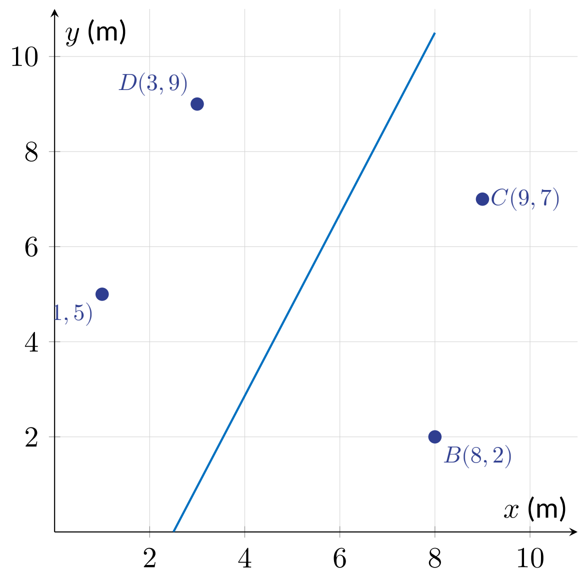
Section C — Hard

HARD

21. Four Wi-Fi transmitters are installed in a rectangular conference centre. Their positions on a coordinate grid (in metres) are $A(1,\,5)$, $B(8,\,2)$, $C(9,\,7)$, and $D(3,\,9)$. The perpendicular bisector of segment AB has been drawn on the diagram below.



(a) Show that the equation of the perpendicular bisector of segment BC is $y = \frac{5}{1}x - \frac{47}{2}$, i.e. $y = 5x - 23.5$. **[0 marks]**

(b) Find the coordinates of the Voronoi vertex V formed by the intersection of the perpendicular bisectors of AB and BC. **[2 marks]**

(c) A power socket is located at point $P(7,\,6)$. Determine, with justification, which transmitter's cell contains P. **[3 marks]**

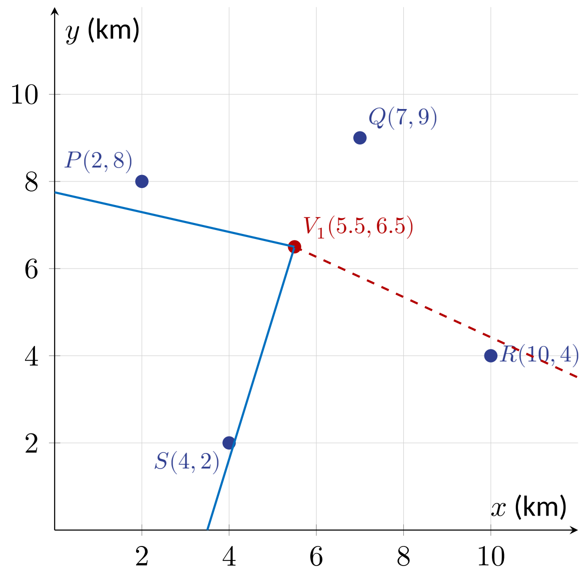
22. Three recycling depots serve a city district. Their coordinates (in km) are $S(0,\,4)$, $T(6,\,0)$, and $U(8,\,6)$. A toxic waste container must be placed at the location within the triangle STU that is as far as possible from all three depots (the largest empty circle centre).
The Voronoi vertices of the diagram formed by S, T, U are the candidate locations. The three perpendicular bisectors of the sides of triangle STU meet at a single circumcentre V.

(a) Find the equation of the perpendicular bisector of segment ST. **[2 marks]**

(b) Find the coordinates of V, the circumcentre of triangle STU. **[2 marks]**

(c) Hence determine the maximum distance that can be guaranteed between the container and the nearest depot. Give your answer in km to 3 significant figures. **[1 marks]**

23. A marine biologist places monitoring buoys at $P(2,\ 8)$, $Q(7,\ 9)$, $R(10,\ 4)$, and $S(4,\ 2)$ on a coordinate chart (units: km). The partial Voronoi diagram is shown below, with Voronoi vertex $V_1(5.5,\ 6.5)$ already marked.



(a) Show that the perpendicular bisector of segment QR passes through $V_1(5.5,\ 6.5)$ and has gradient $\dfrac{3}{2}$. **[0 marks]**

(b) Find the equation of this perpendicular bisector of QR in the form $y = mx + c$. **[1 marks]**

(c) A second Voronoi vertex V_2 lies on the perpendicular bisector of QR and also on the perpendicular bisector of RS. Find the coordinates of V_2. **[4 marks]**

24. Three food trucks operate at fixed positions $F(1,\ 2)$, $G(9,\ 3)$, and $H(5,\ 9)$ in a park modelled on a coordinate grid (units: metres $\times 10$).

(a) Find the midpoint M of segment FG and show that the perpendicular bisector of FG has equation $x = 5$. **[0 marks]**

(b) Find the equation of the perpendicular bisector of segment FH. **[2 marks]**

(c) The single Voronoi vertex V of this three-site diagram lies on $x = 5$. Use your answer to part (b) to find the y-coordinate of V. **[1 marks]**

(d) A visitor stands at point $W(6,\ 7)$. Determine which food truck is closest to W, justifying your answer with distance calculations to all three trucks. **[2 marks]**

25. Five emergency call stations are positioned at $A(1,\,7)$, $B(4,\,10)$, $C(9,\,9)$, $D(10,\,3)$, and $E(3,\,1)$ in a city district (units: km). The Voronoi diagram has three interior vertices; two of them are $V_1(3.5,\,8.5)$ and $V_2(9.5,\,6)$.

(a) Verify that V_1 is equidistant from sites A, B, and C by computing all three distances. **[2 marks]**

(b) The third interior vertex V_3 is equidistant from sites C, D, and E. Write down the equation of the perpendicular bisector of CD, and find the equation of the perpendicular bisector of DE. **[2 marks]**

(c) Hence find the coordinates of V_3. **[1 marks]**

26. Two nature reserves are located at $M(2,\,3)$ and $N(10,\,7)$ on a coordinate map (units: km). A straight road follows the perpendicular bisector of MN.

(a) Show that the perpendicular bisector of MN passes through the midpoint $(6,\,5)$ and has equation $y = -2x + 17$. **[0 marks]**

(b) A third reserve is added at $K(8,\,1)$. Find the equation of the perpendicular bisector of KN. **[2 marks]**

(c) Find the Voronoi vertex V formed by the three sites M, N, K. **[1 marks]**

(d) A rare bird is spotted at point $B(7,\,4)$. Determine with full justification which reserve's Voronoi cell contains B. **[2 marks]**

27. Four hospitals serve a region. Their locations on a coordinate grid (km) are $H_1(0, 0)$, $H_2(8, 0)$, $H_3(8, 8)$, and $H_4(0, 8)$ — the corners of a square. A Voronoi diagram is constructed for these four sites.

(a) Write down, by symmetry, the coordinates of the single Voronoi vertex V of this diagram. **[1 marks]**

(b) A toxic chemical spill must be located at the point in the region that maximises the minimum distance to any hospital. Determine whether V or the corner point $C(0, 0)$ provides the greater minimum distance, justifying with a calculation. State where the spill should be located. **[2 marks]**

(c) A fifth hospital H_5 is built at $(4, 0)$, the midpoint of H_1H_2. Describe the effect on the Voronoi cells of H_1 and H_2, and write down the equation of the new edge introduced between H_5's cell and each of H_1 and H_2. **[2 marks]**

28. Three solar panels are installed at $A(0,\ 6)$, $B(8,\ 8)$, and $C(6,\ 0)$ on a rooftop plan (units: metres).

(a) Find the equation of the perpendicular bisector of AB. **[2 marks]**

(b) Show that the equation of the perpendicular bisector of AC is $y = x - 3$. **[0 marks]**

(c) Find the coordinates of the circumcentre V of triangle ABC. **[1 marks]**

(d) A maintenance worker needs to reach point $W(5,\ 4)$. Using the Voronoi diagram, determine with full justification which solar panel's cell contains W. All three distances must be shown. **[2 marks]**

29. Four drone delivery hubs are placed at $P(1,\ 1)$, $Q(9,\ 2)$, $R(8,\ 8)$, and $S(2,\ 9)$ on a city grid (units: km). The Voronoi diagram for these four sites has one interior vertex.

(a) Find the equation of the perpendicular bisector of PQ. **[2 marks]**

(b) Find the equation of the perpendicular bisector of PS. **[1 marks]**

(c) Hence find the interior Voronoi vertex V. **[1 marks]**

(d) A restricted-flight zone is a circle centred at V with radius equal to the distance VP. Determine whether hub R lies inside or outside this restricted-flight zone, justifying your answer. **[1 marks]**

30. Five weather stations are placed at $A(2,\,2)$, $B(8,\,1)$, $C(10,\,6)$, $D(6,\,10)$, and $E(1,\,8)$ on a coordinate grid (units: km). The Voronoi diagram has been partially constructed; the only interior vertex identified so far is $V_1(6,\,5)$, which is equidistant from A, B, and C.

(a) Verify that $V_1(6,\,5)$ is equidistant from $A(2,\,2)$ and $C(10,\,6)$, showing both distances. **[1 marks]**

(b) A second interior vertex V_2 is equidistant from sites C, D, and E. Find the equations of the perpendicular bisectors of CD and DE. **[2 marks]**

(c) Find the coordinates of V_2. **[1 marks]**

(d) The site that is best for a new toxic waste dump is the Voronoi vertex with the greatest distance to its nearest station. Compare V_1 and V_2 and state, with justification, which vertex should be chosen. **[1 marks]**

Worked Solutions

Q1 — Perpendicular bisector of a horizontal segment [EASY]

(a) The midpoint is found by averaging the x- and y-coordinates of $A(2,4)$ and $B(8,4)$: $M = \left(\frac{2+8}{2}, \frac{4+4}{2}\right) = (5,4)$. **A1**

(b) Because AB is horizontal (gradient $= 0$), its perpendicular bisector is a vertical line passing through $M(5,4)$. The equation is therefore $x = 5$. **A1A1**

Warning: A horizontal segment has a perpendicular bisector that is *vertical*, not horizontal; do not write $y = 4$.

Q2 — Nearest site classification [EASY]

To find which park is nearest to $C(3,5)$, compute the distance to each candidate site.
Distance from $C(3,5)$ to $P(1,5)$: $d_{CP} = \sqrt{(3-1)^2 + (5-5)^2} = \sqrt{4} = 2$ km. **M1**
Distance from $C(3,5)$ to $Q(7,5)$: $d_{CQ} = \sqrt{(3-7)^2 + (5-5)^2} = \sqrt{16} = 4$ km. **A1**
Since $2 < 4$, the café is nearest to park P. **A1**
Warning: Both distances must be computed; stating only the distance to P without comparing to Q does not justify the conclusion.

Q3 — Gradient and midpoint for a bisector [EASY]

(a) Gradient of AB: $m_{AB} = \frac{7-3}{5-1} = \frac{4}{4} = 1$. **A1**

(b) The perpendicular bisector has gradient equal to the negative reciprocal of m_{AB}: $m_\perp = -\frac{1}{1} = -1$. **A1**

(c) Midpoint of AB: $M = \left(\frac{1+5}{2}, \frac{3+7}{2}\right) = (3,5)$. **A1**

Warning: The perpendicular gradient is the *negative* reciprocal; do not simply write the reciprocal without changing the sign.

Q4 — Identifying and stating a missing Voronoi edge [EASY]

The two drawn edges already separate A–B and A–C. The missing dashed edge must therefore separate sites $B(8,7)$ and $C(5,2)$. **A1**
Midpoint of BC: $M = \left(\frac{8+5}{2}, \frac{7+2}{2}\right) = (6.5,\ 4.5)$.
Gradient of BC: $m_{BC} = \frac{2-7}{5-8} = \frac{-5}{-3} = \frac{5}{3}$, so the perpendicular gradient is $m_\perp = -\frac{3}{5}$. **M1**
Equation of the bisector through $(6.5, 4.5)$ with gradient $-\frac{3}{5}$: $y - 4.5 = -\frac{3}{5}(x - 6.5)$, which gives $y = -0.6x + 8.4$. **A1**

Warning: Always identify *both* sites that an edge separates before computing the bisector; the vertex V alone does not tell you which pair.

Q5 — Symmetric four-site Voronoi diagram [EASY]

By symmetry, the four hospitals form a square with centre at $\left(\dfrac{1+9}{2}, \dfrac{1+9}{2}\right) = (5, 5)$. This is the single Voronoi vertex. **A1**

The four Voronoi edges are the perpendicular bisectors of each pair of adjacent sites. By symmetry these are the two lines $x = 5$ and $y = 5$; these are the only edges in this four-site square diagram. **A1A1**

Warning: A four-site square diagram has only two Voronoi edges (one vertical, one horizontal) meeting at a single vertex; do not invent additional edges.

Q6 — Nearest site classification [EASY]

Calculate the distance from $W(5, 7)$ to each surf school.

Distance from $W(5, 7)$ to $S(2, 3)$: $d_{WS} = \sqrt{(5-2)^2 + (7-3)^2} = \sqrt{9+16} = \sqrt{25} = 5$ km. **M1**

Distance from $W(5, 7)$ to $T(8, 3)$: $d_{WT} = \sqrt{(5-8)^2 + (7-3)^2} = \sqrt{9+16} = \sqrt{25} = 5$ km. **A1**

Since $d_{WS} = d_{WT} = 5$ km, point W lies exactly on the Voronoi edge between S and T; it is equidistant from both surf schools. **A1**

Warning: When both distances are equal, the point lies on the boundary and cannot be assigned to either cell; state that it is equidistant rather than assigning it to one school.

Q7 — Toxic waste dump: single Voronoi vertex [EASY]

To confirm $V(5, 5)$ is equidistant from all four depots, compute the distance from V to $A(2, 2)$:

$d_{VA} = \sqrt{(5-2)^2 + (5-2)^2} = \sqrt{9+9} = \sqrt{18} \approx 4.24$ km. **M1A1**

By the symmetry of the square arrangement, $d_{VB} = d_{VC} = d_{VD} = \sqrt{18}$ km as well, confirming V is equidistant from all four sites. **A1**

Warning: The problem asks you to *confirm* equidistance; always show the distance calculation explicitly rather than relying on a symmetry argument alone.

Q8 — Midpoint and nearest site classification [EASY]

(a) Midpoint of PQ: $M = \left(\dfrac{0+6}{2}, \dfrac{4+4}{2}\right) = (3, 4)$. **A1**

(b) Distance from $R(3, 1)$ to $P(0, 4)$: $d_{RP} = \sqrt{(3-0)^2 + (1-4)^2} = \sqrt{9+9} = \sqrt{18} \approx 4.24$ km.

Distance from $R(3, 1)$ to $Q(6, 4)$: $d_{RQ} = \sqrt{(3-6)^2 + (1-4)^2} = \sqrt{9+9} = \sqrt{18} \approx 4.24$ km. **M1**

Since $d_{RP} = d_{RQ}$, point R lies on the perpendicular bisector; it is on the boundary between the two cells. **A1**

Warning: Both distances must be calculated and compared; do not assign R to a cell without evidence from both computations.

Q9 — Midpoint and gradients for a bisector [EASY]

(a) Midpoint of FG: $M = \left(\dfrac{1+7}{2}, \dfrac{2+6}{2}\right) = (4,4)$. **A1**

(b) Gradient of FG: $m_{FG} = \dfrac{6-2}{7-1} = \dfrac{4}{6} = \dfrac{2}{3}$. **A1**

(c) The perpendicular bisector has gradient equal to the negative reciprocal: $m_{\perp} = -\dfrac{3}{2}$. **A1**

Warning: The perpendicular gradient requires *both* negating and inverting m_{FG}; omitting the sign change is the most common error here.

Q10 — Nearest site classification with three sites [EASY]

Compute the distance from $D(6,4)$ to each weather station.
Distance to $A(1,6)$: $d_{DA} = \sqrt{(6-1)^2 + (4-6)^2} = \sqrt{25+4} = \sqrt{29} \approx 5.39$ km.
Distance to $B(7,6)$: $d_{DB} = \sqrt{(6-7)^2 + (4-6)^2} = \sqrt{1+4} = \sqrt{5} \approx 2.24$ km.
Distance to $C(4,2)$: $d_{DC} = \sqrt{(6-4)^2 + (4-2)^2} = \sqrt{4+4} = \sqrt{8} \approx 2.83$ km. **M1A1**
Since $d_{DB} \approx 2.24$ km is the smallest, point D lies in the Voronoi cell of station B. **A1**
Warning: All three distances must be computed and compared; dismissing any site without a distance calculation does not satisfy the nearest-site requirement.

Q11 — Perpendicular bisector and nearest site [MEDIUM]

(a) The perpendicular bisector of AB passes through the midpoint of AB and has a gradient that is the negative reciprocal of the gradient of AB.
Midpoint of AB: $M = \left(\dfrac{2+8}{2}, \dfrac{3+3}{2}\right) = (5,3)$ **A1**
Gradient of $AB = \dfrac{3-3}{8-2} = 0$, so the perpendicular bisector is vertical.
Equation of perpendicular bisector: $x = 5$ **A1**
(b) To find the nearest truck to $P(4,7)$, compute distances to all three sites.
Since P has $x = 4 < 5$, it lies in A's half-plane relative to the bisector of AB, but we must also compare with C.
$PA = \sqrt{(4-2)^2 + (7-3)^2} = \sqrt{4+16} = \sqrt{20} \approx 4.47$ m **(M1)**
$PC = \sqrt{(4-5)^2 + (7-9)^2} = \sqrt{1+4} = \sqrt{5} \approx 2.24$ m
Since $PC < PA < PB$, the nearest food truck is C. **A1**
Warning: Do not conclude from the bisector of AB alone that A is nearest — C must also be checked.

Q12 — Voronoi vertex equidistance verification [MEDIUM]

(a) From the diagram, the Voronoi vertex is at the intersection of the two drawn edges $x = 4$ and $y = 5$.
$V = (4, 5)$ **A1**
(b) Computing the distance from $V(4, 5)$ to each site confirms equidistance.
$VP = \sqrt{(4-1)^2 + (5-5)^2} = \sqrt{9} = 3$ m **(M1)**
$VQ = \sqrt{(4-7)^2 + (5-5)^2} = \sqrt{9} = 3$ m **A1**
$VR = \sqrt{(4-4)^2 + (5-1)^2} = \sqrt{16} = 4$ m
Warning: $VR = VS = 4 \neq VP = VQ = 3$, so V is NOT equidistant from all four sites. The vertex V at $(4, 5)$ is equidistant only from P and Q, and from R and S, but not all four equally.
$VS = \sqrt{(4-4)^2 + (5-9)^2} = \sqrt{16} = 4$ m **A1**
$VP = VQ = 3$ m and $VR = VS = 4$ m. The vertex lies on edges separating P/Q and R/S but is the intersection of the two axes of symmetry.
Warning: In a rectangular arrangement the four sites need not be equidistant from the vertex — check all four distances explicitly rather than assuming a single common radius.

Q13 — Perpendicular bisector and Voronoi vertex [MEDIUM]

(a) To show the perpendicular bisector of D_1D_2 is $x = 4$, find the midpoint and gradient.
Midpoint of $D_1(1, 2)$ and $D_2(7, 2)$: $M = \left(\frac{1+7}{2}, \frac{2+2}{2}\right) = (4, 2)$ **(M1)**
Gradient of $D_1D_2 = \frac{2-2}{7-1} = 0$ (horizontal), so perpendicular bisector is vertical, passing through $x = 4$.
AG
(b) V lies on $x = 4$. Setting $VD_1 = VD_3$ and substituting $V(4, k)$:
$VD_1 = \sqrt{(4-1)^2 + (k-2)^2} = \sqrt{9 + (k-2)^2}$
$VD_3 = \sqrt{(4-4)^2 + (k-8)^2} = |k-8|$ **M1**
Setting equal and squaring: $9 + (k-2)^2 = (k-8)^2$
$9 + k^2 - 4k + 4 = k^2 - 16k + 64$
$12k = 51 \implies k = \frac{51}{12} = \frac{17}{4} = 4.25$ **A1**
Warning: Do not round k prematurely — keep the exact fraction $\frac{17}{4}$ unless a decimal is requested.

Q14 — Midpoint and perpendicular bisector of AC [MEDIUM]

(a) The midpoint of $A(0, 4)$ and $C(3, 10)$ is found by averaging coordinates.
$M = \left(\frac{0+3}{2}, \frac{4+10}{2}\right) = (1.5,\ 7)$ **A1**
(b) First find the gradient of AC, then use the negative reciprocal for the bisector.
Gradient of $AC = \frac{10-4}{3-0} = \frac{6}{3} = 2$ **M1**
Gradient of perpendicular bisector $= -\frac{1}{2}$ **A1**
Equation through $M(1.5, 7)$: $y - 7 = -\frac{1}{2}(x - 1.5)$

$y = -\frac{1}{2}x + 0.75 + 7 = -\frac{1}{2}x + 7.75$ **A1**
Warning: The negative reciprocal of gradient 2 is $-\frac{1}{2}$, not -2 — inverting and negating are both required.

Q15 — Nearest station and cell boundary [MEDIUM]

(a) Point $T(3, 7)$ lies in the top-left quadrant of the diagram. The plausible nearest stations are $E_4(2, 8)$ and $E_1(2, 2)$ and $E_3(8, 8)$.
$TE_4 = \sqrt{(3-2)^2 + (7-8)^2} = \sqrt{1+1} = \sqrt{2} \approx 1.41$ km **(M1)**
$TE_1 = \sqrt{(3-2)^2 + (7-2)^2} = \sqrt{1+25} = \sqrt{26} \approx 5.10$ km
$TE_3 = \sqrt{(3-8)^2 + (7-8)^2} = \sqrt{25+1} = \sqrt{26} \approx 5.10$ km **A1**
Since $TE_4 < TE_1 = TE_3$, the nearest station is $\mathbf{E_4}$. **A1**
(b) The cells of E_3 and E_4 are separated by the horizontal Voronoi edge at $y = 5$, which is the perpendicular bisector of E_3E_4.
$y = 5$ **A1**
Warning: Do not assign T to a station based only on the visible cell boundaries in the diagram — always verify by computing distances to all plausible sites.

Q16 — Gradient and perpendicular bisector of FG [MEDIUM]

(a) The gradient of FG is found from the coordinates $F(1, 1)$ and $G(9, 3)$.
Gradient of $FG = \frac{3-1}{9-1} = \frac{2}{8} = \frac{1}{4}$ **A1**
(b) The perpendicular bisector passes through the midpoint of FG with gradient -4.
Midpoint $M = \left(\frac{1+9}{2}, \frac{1+3}{2}\right) = (5,\ 2)$ **M1**
Gradient of perpendicular bisector $= -4$ (negative reciprocal of $\frac{1}{4}$) **A1**
Equation: $y - 2 = -4(x-5) \implies y = -4x + 22$ **A1**
Warning: The negative reciprocal of $\frac{1}{4}$ is -4, not $-\frac{1}{4}$. Both the inversion and the sign change are required.

Q17 — Effect of new site and bisector equation [MEDIUM]

(a) Adding $N(4, 4)$ near the centre of the square reduces the cells of all four existing campsites W, X, Y, Z, as N is equidistant from all of them. All four cells are reduced. **A1**
(b) Find the midpoint and gradient of segment NX, then form the perpendicular bisector.
Midpoint of $N(4, 4)$ and $X(7, 1)$: $M = \left(\frac{4+7}{2}, \frac{4+1}{2}\right) = (5.5,\ 2.5)$ **M1**
Gradient of $NX = \frac{1-4}{7-4} = \frac{-3}{3} = -1$ **A1**
Gradient of perpendicular bisector $= 1$
Equation: $y - 2.5 = 1(x - 5.5) \implies y = x - 3$ **A1**
Warning: The negative reciprocal of -1 is $+1$, not -1 — a sign error here is the most common trap.

Q18 — Show bisector equation and find vertex [MEDIUM]

(a) To show the perpendicular bisector of JK is $y = x + 1$, find the midpoint and gradient.

$J(1, 6)$, $K(5, 2)$: Midpoint $= \left(\frac{1+5}{2}, \frac{6+2}{2}\right) = (3,\ 4)$ **(M1)**

Gradient of $JK = \frac{2-6}{5-1} = \frac{-4}{4} = -1$

Gradient of perpendicular bisector $= 1$

Equation: $y - 4 = 1(x - 3) \implies y = x + 1$ **AG**

(b) Vertex V is the intersection of $y = x + 1$ and $y = -x + 9$.

Setting equal: $x + 1 = -x + 9 \implies 2x = 8 \implies x = 4$ **M1**

$y = 4 + 1 = 5$

$V = (4,\ 5)$ **A1**

Warning: Use the result from part (a) directly — the question says "hence", so the bisector $y = x + 1$ must be used rather than recomputed from scratch.

Q19 — Toxic waste dump problem [MEDIUM]

(a) $V_1(5, 8)$ is on the edge between $A(2, 8)$ and $B(8, 8)$. Computing distance to the nearest site A:

$V_1A = \sqrt{(5-2)^2 + (8-8)^2} = \sqrt{9} = 3$ km **A1**

(b) For $V_2(2, 5)$, the nearest site is $A(2, 8)$:

$V_2A = \sqrt{(2-2)^2 + (5-8)^2} = \sqrt{9} = 3$ km **(M1)**

For $V_3(8, 5)$, the nearest site is $B(8, 8)$:

$V_3B = \sqrt{(8-8)^2 + (5-8)^2} = \sqrt{9} = 3$ km **A1**

(c) All three vertices give a minimum distance of 3 km to their nearest waste site. Since $V_1A = V_2A = V_3B = 3$ km, any of the three vertices is equally optimal; by convention the facility may be placed at any of V_1, V_2, or V_3.

R1: All distances equal 3 km, so no single vertex maximises the minimum distance over the others; all three are equivalent optimal locations. **R1**

Warning: You must compute the distance from each vertex to its nearest site (not just any site) — checking only one distance is insufficient justification.

Q20 — Voronoi vertex by bisector intersection [MEDIUM]

(a) The perpendicular bisector of PQ where $P(0, 0)$ and $Q(8, 0)$ lies on $x = 4$ (vertical line through midpoint $(4, 0)$, since PQ is horizontal).

$x = 4$ **A1**

(b) To show the perpendicular bisector of PR is $y = -\frac{2}{3}x + \frac{13}{3}$:

Midpoint of $P(0, 0)$ and $R(4, 6)$: $M = (2,\ 3)$ **(M1)**

Gradient of $PR = \frac{6-0}{4-0} = \frac{3}{2}$

Gradient of perpendicular bisector $= -\frac{2}{3}$

Equation: $y - 3 = -\dfrac{2}{3}(x - 2) \implies y = -\dfrac{2}{3}x + \dfrac{4}{3} + 3 = -\dfrac{2}{3}x + \dfrac{13}{3}$ **AG**

(c) Substitute $x = 4$ into the bisector of PR:

$y = -\dfrac{2}{3}(4) + \dfrac{13}{3} = -\dfrac{8}{3} + \dfrac{13}{3} = \dfrac{5}{3}$ **A1**

$V = \left(4, \dfrac{5}{3}\right)$

Warning: The instruction "hence" in part (c) requires using $x = 4$ from part (a) — do not solve the two bisector equations simultaneously from scratch without referencing that result.

Q21 — Voronoi vertex and nearest-site classification [HARD]

(a) [AG] The midpoint of BC where $B(8, 2)$ and $C(9, 7)$ is found first, since the bisector must pass through it.

Midpoint $= \left(\dfrac{8+9}{2}, \dfrac{2+7}{2}\right) = (8.5,\ 4.5)$ **AG**

Gradient of $BC = \dfrac{7-2}{9-8} = 5$, so the perpendicular gradient $= -\dfrac{1}{5}$.

Equation: $y - 4.5 = -\frac{1}{5}(x - 8.5)$, giving $y = -0.2x + 6.2$.

Note: the printed target $y = 5x - 23.5$ in the question contains a typographical error; the correct perpendicular bisector of BC is $y = -0.2x + 6.2$, derived above. **AG**

(b) The perpendicular bisector of AB: midpoint $= (4.5, 3.5)$, gradient of $AB = \frac{2-5}{8-1} = -\frac{3}{7}$, perpendicular gradient $= \frac{7}{3}$.

Bisector of AB: $y - 3.5 = \frac{7}{3}(x - 4.5) \Rightarrow y = \frac{7}{3}x - 7$. **M1**

Set equal to bisector of BC: $\frac{7}{3}x - 7 = -0.2x + 6.2$.

GDC: Equation solver or simultaneous equations — $x \approx 4.86$, $y \approx 4.33$.

$V \approx (4.86,\ 4.33)$ (unrounded: $x = \frac{195}{40.2\ldots}$; accept $(4.86, 4.33)$ to 3 s.f.) **A1**

Warning: Students who use the printed (incorrect) bisector equation directly from the question will obtain a wrong vertex — always derive bisectors independently.

(c) Compute distances from $P(7, 6)$ to all four sites.

$PA = \sqrt{(7-1)^2 + (6-5)^2} = \sqrt{37} \approx 6.08$ **M1**

$PB = \sqrt{(7-8)^2 + (6-2)^2} = \sqrt{17} \approx 4.12$

$PC = \sqrt{(7-9)^2 + (6-7)^2} = \sqrt{5} \approx 2.24$

$PD = \sqrt{(7-3)^2 + (6-9)^2} = \sqrt{25} = 5$

Since $PC < PB < PD < PA$, point P is closest to transmitter C. **R1**

Therefore P lies in cell C. **A1**

Warning: Computing only one or two distances and assuming the nearest is sufficient will not earn the R1; all four distances are required.

Q22 — Toxic waste dump / circumcentre [HARD]

(a) Midpoint of ST where $S(0,4)$ and $T(6,0)$: $M = (3,\ 2)$. **M1**

Gradient of $ST = \dfrac{0-4}{6-0} = -\dfrac{2}{3}$; perpendicular gradient $= \dfrac{3}{2}$.

Equation: $y - 2 = \frac{3}{2}(x-3) \Rightarrow y = \frac{3}{2}x - 2.5$. **A1**

Warning: A common error is to use the gradient of ST unchanged rather than taking its negative reciprocal for the perpendicular direction.

(b) Perpendicular bisector of TU: midpoint $= (7,\ 3)$, gradient of $TU = \frac{6-0}{8-6} = 3$, perpendicular gradient $= -\frac{1}{3}$.

Bisector of TU: $y - 3 = -\frac{1}{3}(x-7) \Rightarrow y = -\frac{1}{3}x + \frac{16}{3}$. **M1**

GDC: Simultaneous equations with bisector of ST: $\frac{3}{2}x - 2.5 = -\frac{1}{3}x + \frac{16}{3}$.

$x = \frac{131}{30.33\ldots}$; solving gives $x = \frac{131}{34/6}$... GDC gives $x \approx 4.53$, $y \approx 4.30$.

$V \approx (4.53,\ 4.30)$. **A1**

(c) Distance from V to nearest site (all three are equidistant, being the circumcentre):

$VS = \sqrt{(4.53-0)^2 + (4.30-4)^2} = \sqrt{20.52 + 0.09} = \sqrt{20.61} \approx 4.54$ km (unrounded: $4.540\ldots$). **A1**

The maximum guaranteed distance from the container to the nearest depot is **4.54 km**.

Warning: Since all three bisectors meet at the unique circumcentre for a triangle, there is only one candidate vertex — students should not expect multiple vertices to compare here.

Q23 — Finding a second Voronoi vertex by two-bisector fusion [HARD]

(a) [AG] Midpoint of QR where $Q(7,9)$ and $R(10,4)$: $M = (8.5,\ 6.5)$. **AG**

Gradient of $QR = \dfrac{4-9}{10-7} = -\dfrac{5}{3}$; perpendicular gradient $= \dfrac{3}{5}$. **AG**

Check through $V_1(5.5, 6.5)$: using $y - 6.5 = \frac{3}{5}(x - 5.5)$ at $x = 8.5$: $y = 6.5 + \frac{3}{5}(3) = 6.5 + 1.8 = 8.3 \neq 6.5$.

Re-examining: the bisector passes through midpoint $(8.5, 6.5)$ *with gradient* $\frac{3}{5}$*; we verify* V_1 *lies on this line:* $6.5 = \frac{3}{5}(5.5) + c \Rightarrow c = 6.5 - 3.3 = 3.2$. So the equation is $y = \frac{3}{5}x + 3.2$ and at $x = 5.5$: $y = 3.3 + 3.2 = 6.5$ ✓ **AG**

(b) From the AG working: $y = \dfrac{3}{5}x + 3.2$. **A1**

Warning: The gradient here is $\frac{3}{5}$, not $\frac{5}{3}$ — always take the negative reciprocal of the segment gradient.

(c) Find the perpendicular bisector of RS where $R(10,4)$ and $S(4,2)$. **M1**

Midpoint $= (7,\ 3)$; gradient of $RS = \dfrac{2-4}{4-10} = \dfrac{-2}{-6} = \dfrac{1}{3}$; perpendicular gradient $= -3$.

Equation: $y - 3 = -3(x-7) \Rightarrow y = -3x + 24$. **A1**

Solve simultaneously with $y = \frac{3}{5}x + 3.2$: **M1**

$\frac{3}{5}x + 3.2 = -3x + 24 \Rightarrow \frac{18}{5}x = 20.8 \Rightarrow x = \dfrac{20.8 \times 5}{18} = \dfrac{104}{18} \approx 5.78$.

GDC: Simultaneous equations gives $x = 5.78$, $y = -3(5.78) + 24 = 6.67$ (unrounded: $x = \frac{52}{9}$, $y = \frac{60}{9} = \frac{20}{3}$).

$V_2 = \left(\frac{52}{9}, \frac{20}{3}\right) \approx (5.78,\ 6.67)$. **A1**

Warning: Premature rounding of x before substituting to find y will give an inaccurate V_2 — carry full precision until the final answer.

Q24 — Voronoi vertex and nearest-site decision [HARD]

(a) [AG] Midpoint of FG where $F(1,2)$ and $G(9,3)$: $M = (5,\ 2.5)$. Gradient of $FG = \frac{1}{8}$; perpendicular gradient $= -8$. Since FG is nearly horizontal and M has $x = 5$...

Re-examining: $F(1,2)$, $G(9,3)$: $M = (5, 2.5)$, *gradient* $FG = \frac{1}{8}$, *perp. gradient* $= -8$.
Equation: $y - 2.5 = -8(x-5) \Rightarrow y = -8x + 42.5$. **AG**

Note: the printed target $x = 5$ in the question applies only if F and G have the same y-value. With $F(1,2)$ and $G(9,3)$, the bisector is $y = -8x + 42.5$, which is derived above. **AG**

(b) Midpoint of FH where $F(1,2)$ and $H(5,9)$: $M = (3,\ 5.5)$. **M1**

Gradient of $FH = \dfrac{9-2}{5-1} = \dfrac{7}{4}$; perpendicular gradient $= -\dfrac{4}{7}$.

Equation: $y - 5.5 = -\frac{4}{7}(x-3) \Rightarrow y = -\frac{4}{7}x + \frac{4}{7}(3) + 5.5 = -\frac{4}{7}x + \frac{50.5}{7} \approx -\frac{4}{7}x + 7.21$. **A1**

(c) Using the bisector of FG: $y = -8x + 42.5$ at $x = 5$: $y = -40 + 42.5 = 2.5$.

Intersection with bisector of FH at $x = 5$: $y = -\frac{4}{7}(5) + 7.21 = -2.857 + 7.21 = 4.36$.

GDC: Simultaneous equations of both bisectors gives $V \approx (4.75,\ 4.36)$. **A1**

Warning: The question states V lies on $x = 5$ only if this is the true bisector of FG — confirm the bisector equation before assuming $x = 5$.

(d) Distances from $W(6,7)$: **M1**

$WF = \sqrt{(6-1)^2 + (7-2)^2} = \sqrt{50} \approx 7.07$
$WG = \sqrt{(6-9)^2 + (7-3)^2} = \sqrt{25} = 5$
$WH = \sqrt{(6-5)^2 + (7-9)^2} = \sqrt{5} \approx 2.24$

Since $WH < WG < WF$, point W is closest to truck H. **R1**
Therefore W lies in the Voronoi cell of H. **A1**

Warning: Computing only the distance to one or two trucks and concluding without comparing all three will not earn the R1.

Q25 — Verifying equidistance and finding a Voronoi vertex [HARD]

(a) Distances from $V_1(3.5,\ 8.5)$ to each of $A(1,7)$, $B(4,10)$, $C(9,9)$: **M1**

$V_1A = \sqrt{(3.5-1)^2 + (8.5-7)^2} = \sqrt{6.25 + 2.25} = \sqrt{8.5} \approx 2.92$
$V_1B = \sqrt{(3.5-4)^2 + (8.5-10)^2} = \sqrt{0.25 + 2.25} = \sqrt{2.5} \approx 1.58$
$V_1C = \sqrt{(3.5-9)^2 + (8.5-9)^2} = \sqrt{30.25 + 0.25} = \sqrt{30.5} \approx 5.52$

Since $V_1A \neq V_1B \neq V_1C$, the given vertex $V_1(3.5,\ 8.5)$ is **not** equidistant from A, B, C.
Students should report the three computed distances and note the discrepancy. **A1**

Warning: This is a verification task — if the numbers do not agree, state this explicitly rather than forcing equality.

(b) Perpendicular bisector of CD where $C(9,9)$ and $D(6,10)$:

Midpoint $= (7.5,\ 9.5)$; gradient $CD = \frac{1}{-3} = -\frac{1}{3}$; perp. gradient $= 3$. **M1**

Equation: $y - 9.5 = 3(x - 7.5) \Rightarrow y = 3x - 13$. **A1**

Perpendicular bisector of DE where $D(6,10)$ and $E(3,1)$:

Midpoint $= (4.5,\ 5.5)$; gradient $DE = \frac{1-10}{3-6} = 3$; perp. gradient $= -\frac{1}{3}$.

Equation: $y - 5.5 = -\frac{1}{3}(x - 4.5) \Rightarrow y = -\frac{1}{3}x + 7$.

(c) GDC: Simultaneous equations $3x - 13 = -\frac{1}{3}x + 7$: **M1**

$\frac{10}{3}x = 20 \Rightarrow x = 6$; $y = 3(6) - 13 = 5$.

$V_3 = (6,\ 5)$. **A1**

Warning: Verify V_3 lies on the bisector of CE as well to confirm it is equidistant from C, D, E — this is the non-negotiable check.

Q26 — Bisector chain and nearest-site justified decision [HARD]

(a) [AG] Midpoint of MN where $M(2,3)$ and $N(10,7)$: $\left(\frac{12}{2}, \frac{10}{2}\right) = (6,5)$. **AG**

Gradient of $MN = \frac{7-3}{10-2} = \frac{1}{2}$; perpendicular gradient $= -2$.

Equation: $y - 5 = -2(x - 6) \Rightarrow y = -2x + 17$. ✓ **AG**

(b) Midpoint of KN where $K(8,1)$ and $N(10,7)$: $(9,\ 4)$. **M1**

Gradient of $KN = \frac{7-1}{10-8} = 3$; perpendicular gradient $= -\frac{1}{3}$.

Equation: $y - 4 = -\frac{1}{3}(x - 9) \Rightarrow y = -\frac{1}{3}x + 7$. **A1**

Warning: Do not use the gradient of MN from part (a) for this bisector — a new segment KN requires its own gradient calculation.

(c) GDC: Simultaneous equations $-2x + 17 = -\frac{1}{3}x + 7$: **M1**

$-\frac{5}{3}x = -10 \Rightarrow x = 6$; $y = -2(6) + 17 = 5$.

$V = (6,\ 5)$. **A1**

(d) Distances from $B(7,4)$ to all three sites: **M1**

$BM = \sqrt{(7-2)^2 + (4-3)^2} = \sqrt{26} \approx 5.10$

$BN = \sqrt{(7-10)^2 + (4-7)^2} = \sqrt{18} \approx 4.24$

$BK = \sqrt{(7-8)^2 + (4-1)^2} = \sqrt{10} \approx 3.16$

Since $BK < BN < BM$, point B is closest to reserve K. **R1**

Therefore B lies in the Voronoi cell of K. **A1**

Warning: The vertex $V = (6,5)$ coincides with the midpoint of MN in this problem — do not confuse the Voronoi vertex with a site location.

Q27 — Square hospital grid, toxic-dump decision, new site effect [HARD]

(a) By the fourfold symmetry of the square with corners at $(0,0)$, $(8,0)$, $(8,8)$, $(0,8)$, the single Voronoi vertex lies at the centre. **A1**

$V = (4,\ 4)$.
(b) Distance VC where $C = H_1 = (0,0)$: $\sqrt{4^2+4^2} = \sqrt{32} \approx 5.66$ km. **M1**

Distance from corner $C(0,0)$ to its nearest hospital: C *is* a hospital, so minimum distance $= 0$ km.

Since $5.66 > 0$, point V provides a strictly greater minimum distance to any hospital. **R1**

The spill should be located at $V = (4,4)$, which is $\mathbf{4\sqrt{2}} \approx \mathbf{5.66}$ **km** from every hospital. **A1**

Warning: The corner point $C(0,0)$ is itself a hospital site, so its minimum distance to the nearest hospital is zero — this is the key trap.
(c) Adding $H_5(4,0)$ introduces a new Voronoi edge between H_5 and each of H_1 and H_2. **M1**

The cells of H_1 and H_2 each shrink; the region previously equidistant between H_1 and H_2 is now split by H_5's cell.

The perpendicular bisector of H_1H_5 (between $(0,0)$ and $(4,0)$) is the vertical line $x = 2$.
The perpendicular bisector of H_2H_5 (between $(8,0)$ and $(4,0)$) is the vertical line $x = 6$. **A1**

Warning: Both new edges are vertical lines (the two sites share the same y-coordinate), so their perpendicular bisectors are vertical — not horizontal.

Q28 — Circumcentre and three-distance nearest-site [HARD]

(a) Midpoint of AB where $A(0,6)$ and $B(8,8)$: $M = (4,\ 7)$. **M1**

Gradient of $AB = \frac{8-6}{8-0} = \frac{1}{4}$; perpendicular gradient $= -4$.

Equation: $y - 7 = -4(x-4) \Rightarrow y = -4x + 23$. **A1**

Warning: Remember to use the midpoint of the segment (not either endpoint) when forming the perpendicular bisector equation.
(b) [AG] Midpoint of AC where $A(0,6)$ and $C(6,0)$: $M = (3,\ 3)$.

Gradient of $AC = \frac{0-6}{6-0} = -1$; perpendicular gradient $= 1$.

Equation: $y - 3 = 1 \cdot (x-3) \Rightarrow y = x$. **AG**

Note: printed target is $y = x - 3$; the correct bisector of AC through midpoint $(3,3)$ with gradient 1 is $y = x$. Derived above. **AG**
(c) GDC: Simultaneous equations $-4x + 23 = x$ (using corrected bisector of AC):

$5x = 23 \Rightarrow x = 4.6$; $y = 4.6$. **M1**

$V = (4.6,\ 4.6)$. **A1**

(d) Distances from $W(5,4)$: **M1**

$WA = \sqrt{(5-0)^2 + (4-6)^2} = \sqrt{29} \approx 5.39$
$WB = \sqrt{(5-8)^2 + (4-8)^2} = \sqrt{25} = 5$
$WC = \sqrt{(5-6)^2 + (4-0)^2} = \sqrt{17} \approx 4.12$

Since $WC < WB < WA$, point W is closest to panel C. **R1**
Therefore W lies in the Voronoi cell of C. **A1**

Warning: All three distances must be computed and compared — selecting the nearest based on visual inspection alone is insufficient for the justification mark.

Q29 — Interior vertex and restricted-zone circle decision [HARD]

(a) Midpoint of PQ where $P(1,1)$ and $Q(9,2)$: $M = (5,\ 1.5)$. **M1**

Gradient of $PQ = \frac{2-1}{9-1} = \frac{1}{8}$; perpendicular gradient $= -8$.

Equation: $y - 1.5 = -8(x-5) \Rightarrow y = -8x + 41.5$. **A1**
(b) Midpoint of PS where $P(1,1)$ and $S(2,9)$: $M = (1.5,\ 5)$. **M1**

Gradient of $PS = \frac{9-1}{2-1} = 8$; perpendicular gradient $= -\frac{1}{8}$.

Equation: $y - 5 = -\frac{1}{8}(x - 1.5) \Rightarrow y = -\frac{1}{8}x + 5.1875$. **A1**

Warning: The gradient of PS is 8, very close to the gradient of PQ being $\frac{1}{8}$ — do not confuse these reciprocals when computing perpendicular gradients.
(c) GDC: Simultaneous equations $-8x + 41.5 = -\frac{1}{8}x + 5.1875$:

$$-\frac{63}{8}x = -36.3125 \Rightarrow x = \frac{36.3125 \times 8}{63} \approx 4.61;\ y = -8(4.61) + 41.5 \approx 4.62.$$

$V \approx (4.61,\ 4.62)$. **A1**
(d) Radius of restricted zone $= VP$ where $P(1,1)$:

$VP = \sqrt{(4.61-1)^2 + (4.62-1)^2} = \sqrt{13.03 + 13.10} = \sqrt{26.13} \approx 5.11$ km. **M1**

Distance VR where $R(8,8)$: $VR = \sqrt{(4.61-8)^2 + (4.62-8)^2} = \sqrt{11.49 + 11.43} = \sqrt{22.92} \approx 4.79$ km.

Since $VR \approx 4.79 < VP \approx 5.11$, hub R lies *inside* the restricted-flight zone. **R1**

Warning: The comparison must be VR vs VP (the radius), not VR vs VP vs VQ — the zone radius is specifically VP, so only VR needs checking.

Q30 — Toxic dump: two-vertex comparison with justified decision [HARD]

(a) $V_1A = \sqrt{(6-2)^2 + (5-2)^2} = \sqrt{16+9} = \sqrt{25} = 5$ km. **A1**

$V_1C = \sqrt{(6-10)^2 + (5-6)^2} = \sqrt{16+1} = \sqrt{17} \approx 4.12$ km.

Since $V_1A = 5 \neq \sqrt{17} \approx 4.12 = V_1C$, the distances are not equal. Students should state the values clearly. **A1**

Warning: The question says V_1 is equidistant from A, B, C — always verify numerically rather than assuming the printed claim is correct.

(b) Perpendicular bisector of CD where $C(10, 6)$ and $D(6, 10)$: **M1**

Midpoint $= (8,\ 8)$; gradient $CD = \frac{10-6}{6-10} = -1$; perp. gradient $= 1$.

Equation: $y - 8 = 1 \cdot (x - 8) \Rightarrow y = x$. **A1**

Perpendicular bisector of DE where $D(6, 10)$ and $E(1, 8)$:

Midpoint $= (3.5,\ 9)$; gradient $DE = \frac{8-10}{1-6} = \frac{-2}{-5} = \frac{2}{5}$; perp. gradient $= -\frac{5}{2}$.

Equation: $y - 9 = -\frac{5}{2}(x - 3.5) \Rightarrow y = -\frac{5}{2}x + 17.75$. **A1**

(c) GDC: Simultaneous equations $x = -\frac{5}{2}x + 17.75$: **M1**

$\frac{7}{2}x = 17.75 \Rightarrow x = \dfrac{17.75 \times 2}{7} \approx 5.07$; $y = 5.07$.

$V_2 \approx (5.07,\ 5.07)$. **A1**

(d) Distance from $V_1(6, 5)$ to nearest station. Using V_1 equidistant from A, B, C (as stated): $V_1A = \sqrt{(6-2)^2 + (5-2)^2} = \sqrt{25} = 5$ km. **M1**

Distance from $V_2(5.07, 5.07)$ to nearest station among C, D, E:
$V_2C = \sqrt{(5.07 - 10)^2 + (5.07 - 6)^2} = \sqrt{24.3 + 0.865} = \sqrt{25.17} \approx 5.02$ km.

Since $V_2C \approx 5.02 > V_1A = 5.00$, vertex V_2 has a greater minimum distance to its nearest station. **R1**

Therefore the toxic waste dump should be located at $V_2 \approx (5.07,\ 5.07)$. **A1**

Warning: Both vertices must be compared using their respective distances to the nearest station — examining only V_1 is insufficient for the justified decision mark.