IB Math AI SL — Topic 4

Probability Distributions

Discrete Probability Distributions · Expected Values

Difficulty	EASY	MEDIUM	HARD
Questions	10	10	10
Total Marks	30	40	50

Name:______________________________ Date:_______________

Score:__________/120

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Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. A graphic display calculator is required throughout; support GDC answers with the values entered and the feature used.

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Key Formulae

- **Sum of all probabilities:** $\sum_{\text{all } x} P(X = x) = 1$
- **Expected value (mean):** $E(X) = \sum_{\text{all } x} x \cdot P(X = x)$
- **Expected value of a function:** $E(g(X)) = \sum_{\text{all } x} g(x) \cdot P(X = x)$
- **Linear transformation of expectation:** $E(aX + b) = a\,E(X) + b$
- **Complementary probability:** $P(X \geq a) = 1 - P(X \leq a - 1)$
- **Conditional probability:** $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
- **Fair game condition:** expected net winnings $= 0$, i.e. $E(\text{winnings}) =$ cost of play

GDC Tips

- **Storing a distribution and computing $E(X)$:** Enter x-values in `List 1` and $P(X = x)$ values in `List 2`; go to `Statistics` $\to$ `1-Var Stats`, set `XList = L1`, `FreqList = L2`; read off $\bar{x}$, which equals $E(X)$.
- **Checking the sum-to-1 condition:** With probabilities in `List 2`, use `List` $\to$ `Math` $\to$ `sum(L2)` and confirm the result equals 1 before proceeding.
- **Solving for an unknown k:** Enter the sum-to-1 equation symbolically in `Algebra` $\to$ `Solve` (e.g. `solve(0.1 + k + 0.3 + 2k = 1, k)`) to find k without manual algebra.
- **Computing $E(g(X))$:** Define a new list `L3 = g(L1)` and then `L4 = L3 * L2`; compute `sum(L4)` to obtain $E(g(X))$ directly.
- **Evaluating cumulative probabilities:** For $P(X \leq a)$, highlight relevant cells in `List 2` and use `sum`; for $P(X \geq a)$ compute $1 -$ `sum` of the preceding cells.
- **Simultaneous equations for two unknowns:** If sum-to-1 and a given $E(X)$ each yield an equation in p and q, use `Algebra` $\to$ `Solve System of Equations` $\to$ `2 equations, 2 unknowns`.

Common Mistakes to Avoid

1. **Forgetting to use sum-to-1 first.** Whenever a probability table contains an unknown k, the very first line of working must set $\sum P(X = x) = 1$ and solve for k. Every subsequent probability read from the table depends on this value; skipping this step invalidates all following parts.
2. **Using an unsolved k in $E(X)$.** Students sometimes substitute the symbolic expression for k into $E(X) = \sum x \cdot P(X = x)$ and then forget to replace k with its numerical value. Always substitute the fully evaluated k before multiplying and summing.
3. **Confusing $E(X)$ with the most likely value.** $E(X)$ is the long-run average, computed as a weighted sum; it need not equal any value in the table and is not necessarily the x with the highest probability. Never read $E(X)$ directly from the table without calculating.
4. **Misreading the fair-game condition.** A game is fair when $E(\text{net gain}) = 0$, meaning the entry fee equals $E(\text{prize})$. A common error is setting the fee equal to $E(X)$ without subtracting the fee itself from each outcome first, or equivalently setting $E(\text{prize}) - \text{fee} = 0$ but then forgetting to check the sign carefully.
5. **Scaling expected winnings incorrectly over N plays.** If $E(\text{net gain per game}) = -0.40$, then over 50 games the expected total loss is $50 \times (-0.40) = -\$20.00$, not $-\$0.40$. Always state clearly whether a computed expectation is per game or total, and multiply explicitly by N.
6. **Accepting a negative or out-of-range probability.** After solving for k, always verify that every individual probability satisfies $0 \leq P(X = x) \leq 1$. If any value of k produces a negative probability or a value exceeding 1, that solution must be rejected and the setup rechecked before continuing.

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Section A — Easy

EASY

1. The discrete random variable X has the following probability distribution, where k is a constant.

x	1	2	3	4
$P(X = x)$	0.2	k	0.35	0.1

Find the value of k. **[3 marks]**

2. A bag contains coloured counters. The discrete random variable Y represents the number shown on a randomly selected card, with probability distribution given below, where k is a constant.

y	0	1	2
$P(Y = y)$	0.4	k	0.25

(a) Find the value of k. **[1 marks]**

(b) Find E(Y). **[2 marks]**

3. The discrete random variable W takes values 1, 2 and 3 only. It is known that $P(W = 1) = 0.5$, $P(W = 2) = 0.3$ and $P(W = 3) = 0.2$.
Find E(W). **[3 marks]**

4. A spinner at a school fair is divided into four sectors labelled 1, 2, 3 and 4. The probability of landing on each sector is shown in the table below, where k is a constant.

x	1	2	3	4
$P(X = x)$	k	0.3	$2k$	0.1

(a) Find the value of k. **[2 marks]**

(b) Write down $P(X = 3)$. **[1 marks]**

5. The discrete random variable X represents the number of text messages a student sends in one hour. The probability distribution of X is given by

$$P(X = x) = \frac{x}{10}, \quad x = 1, 2, 3, 4.$$

(a) Show that the probabilities sum to 1. **[1 marks]**

(b) Find E(X). **[2 marks]**

6. A discrete random variable X has the following probability distribution.

x	0	1	2	3	4
$P(X = x)$	0.05	0.20	0.40	0.25	0.10

(a) Find E(X). **[2 marks]**

(b) Write down $P(X \geq 3)$. **[1 marks]**

7. A prize wheel at a charity event stops on one of three amounts: $0, $5 or $10. The respective probabilities are 0.60, 0.30 and 0.10.
Let X represent the prize amount, in dollars, won on a single spin.
Find E(X). **[3 marks]**

8. The discrete random variable T takes the values 2, 4 and 6 only, with the following probability distribution, where k is a constant.

t	2	4	6
$P(T = t)$	0.3	k	0.2

(a) Find the value of k. **[1 marks]**

(b) Find E(T). **[2 marks]**

9. The discrete random variable X represents the score on a biased four-sided die with faces numbered 1 to 4. The probabilities are $P(X = 1) = 0.1$, $P(X = 2) = 0.4$, $P(X = 3) = 0.3$ and $P(X = 4) = 0.2$.
Find E(X). **[3 marks]**

10. The discrete random variable M has probability distribution given by

$$P(M = m) = \frac{k}{m}, \quad m = 1, 2, 4.$$

(a) Find the value of k. **[2 marks]**

(b) Find E(M). **[1 marks]**

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Section B — Medium

MEDIUM

11. A school carnival game involves spinning a wheel. The number of tokens, X, a player wins in one spin has the probability distribution shown in the table below.

x	0	1	3	5
$P(X=x)$	0.40	k	$2k$	0.10

(a) Find the value of k. **[2 marks]**

(b) Find $\mathrm{E}(X)$, the expected number of tokens won per spin. **[2 marks]**

12. A café loyalty card gives customers a random reward each visit. The monetary reward, R (in dollars), has the following probability distribution, where a and b are constants.

r	0	2	5	10
$P(R = r)$	a	0.35	b	0.05

Given that E$(R) = 1.65$, find the values of a and b. **[4 marks]**

13. A charity raffle sells 200 tickets at \$3 each. There is one prize of \$150, two prizes of \$50, and five prizes of \$10. A player buys one ticket. Let W be the net winnings (prize money minus ticket cost) for the player.

(a) Write down all possible values of W. **[1 marks]**

(b) Find E(W). **[2 marks]**

(c) Determine whether buying a ticket is a fair game. Justify your answer. **[1 marks]**

14. A discrete random variable X has probability distribution defined by

$$P(X = x) = k(x^2 + 1), \quad x \in \{0, 1, 2, 3\}.$$

(a) Show that $k = \dfrac{1}{19}$. **[2 marks]**

(b) Find E(X). **[2 marks]**

15. Two fair four-sided dice, each numbered 1 to 4, are rolled. Let M be the minimum of the two values shown.

(a) Complete the probability distribution table for M. **[2 marks]**

m	1	2	3	4
$P(M = m)$	$\frac{7}{16}$			$\frac{1}{16}$

(b) Find E(M). **[2 marks]**

16. A game at a school fair costs \$4 to play. A player rolls a standard fair six-sided die. If the result is a 6, the player wins \$15. If the result is a 4 or 5, the player wins \$5. Otherwise, the player wins nothing. Let G be the player's net gain (winnings minus entry fee) in one play.

(a) Find E(G). **[3 marks]**

(b) A student plans to play this game 30 times. Find the student's expected total net gain. **[1 marks]**

17. The number of goals, X, scored by a school football team in a match has the probability distribution shown.

x	0	1	2	3	4
$P(X = x)$	0.15	0.30	k	0.15	0.05

(a) Find the value of k. **[1 marks]**

(b) Find E(X). **[2 marks]**

(c) Interpret the value of E(X) in context. **[1 marks]**

18. A spinner is divided into four sectors labelled 1, 2, 3, and 4. The probability of landing on sector x is proportional to x (i.e. $P(X = x) = cx$ for some constant c).

(a) Show that $c = 0.1$. **[1 marks]**

(b) Find $P(X \geq 3)$. **[1 marks]**

(c) Find E(X). **[2 marks]**

19. A bag contains 4 red and 6 blue counters. Two counters are drawn **without replacement**. Let Y be the number of red counters drawn.

(a) Find the probability distribution of Y, showing all probabilities as fractions. **[2 marks]**

(b) Find E(Y). **[2 marks]**

20. A discrete random variable X has the probability distribution given below, where p and q are positive constants.

x	1	2	4	8
$P(X = x)$	p	0.20	q	0.10

Given that E$(X) = 3.2$, find the values of p and q. **[4 marks]**

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Section C — Hard

HARD

21. A board game uses a spinner with four sectors numbered 1, 2, 3 and 4. The probability distribution of the score X on a single spin is given in the table below.

x	1	2	3	4
$P(X = x)$	$2k$	$k + 0.1$	$3k$	0.2

(a) Show that $k = 0.1$. **[2 marks]**

(b) Find E(X). **[2 marks]**

(c) A player pays \$1.50 to spin once and receives \$$X$ in return. Determine whether the game is fair. Justify your answer. **[1 marks]**

22. A discrete random variable Y has the probability distribution defined by

$$P(Y = y) = c(y^2 + 1), \quad y \in \{0, 1, 2, 3\}.$$

(a) Find the value of c. **[2 marks]**

(b) Find $P(Y \geq 2)$. **[1 marks]**

(c) Find E(Y). **[2 marks]**

23. A bag contains 3 red and 2 blue tokens. Two tokens are drawn **without replacement**. The discrete random variable W represents the number of red tokens drawn.

(a) Show that $P(W = 2) = \dfrac{3}{10}$. **[2 marks]**

(b) Complete the probability distribution of W and find E(W). **[3 marks]**

24. Two fair six-sided dice are rolled. The random variable M is the **minimum** of the two scores shown.

(a) Show that $P(M=1)=\dfrac{11}{36}$. **[2 marks]**

(b) Write down $P(M=6)$. **[1 marks]**

(c) Find E(M). **[2 marks]**

25. The probability distribution of a discrete random variable X is given in the table below, where a and b are positive constants.

x	0	1	2	3
$P(X=x)$	a	0.3	b	0.1

Given that E$(X)=1.2$, find the values of a and b. **[5 marks]**

26. At a school fair, a game involves rolling a single fair six-sided die. A player wins \$5 if the die shows a 6, wins \$2 if the die shows a 4 or 5, and wins nothing otherwise. The cost to play is \$$c$.

(a) Find E(W), where W is the amount won (excluding the entry fee). **[2 marks]**

(b) Find the value of c that makes the game fair. **[1 marks]**

(c) A player intends to play this game 30 times. Using your value of c from part (b), find the expected total profit (or loss) over 30 plays. **[2 marks]**

27. A discrete random variable X has the probability distribution defined by

$$P(X = x) = k(4 - x), \quad x \in \{0, 1, 2, 3\}.$$

(a) Find the value of k. **[1 marks]**

(b) Find E(X). **[2 marks]**

(c) Given that a player pays \$0.80 per game and receives \$$X$ in return, find the player's expected loss per game. Determine whether a rational player should participate. Justify your answer. **[2 marks]**

28. A raffle sells 50 tickets at \$4 each. There is one first prize of \$80, two second prizes of \$20 each, and no other prizes. A player buys one ticket. Let G be the net gain to the player (prize money received minus ticket cost).

(a) Write down the probability distribution of G. **[2 marks]**

(b) Find $\mathrm{E}(G)$. **[2 marks]**

(c) Determine, with justification, whether buying a ticket is a financially rational decision. **[1 marks]**

29. The discrete random variable X has the following probability distribution, where p and q are positive constants with $p > q$.

x	-1	0	1	2
$P(X = x)$	p	0.25	q	0.05

Given that $\mathrm{E}(X) = 0$, find p and q. **[5 marks]**

30. A factory produces light bulbs. The number of defective bulbs D in a randomly selected box of 3 bulbs has the probability distribution shown below, where m is a constant.

d	0	1	2	3
$P(D=d)$	0.5	m	0.1	0.02

(a) Find the value of m. **[1 marks]**

(b) Find E(D). **[2 marks]**

(c) A quality inspector tests 5 independent boxes and considers the process acceptable if the expected total number of defective bulbs across all 5 boxes is less than 2. Determine whether the process is acceptable. Justify your answer. **[2 marks]**

Worked Solutions

Q1 — Find k from sum-to-one [EASY]

Because all probabilities in a distribution must sum to exactly 1, we write the equation and solve for k.

$0.2 + k + 0.35 + 0.1 = 1$ **M1**

$k = 1 - 0.65 = 0.35$ **A1**

Warning: Always use sum-to-one as the very first step whenever a probability table contains an unknown; do not attempt to read off k by comparing individual values.

Q2 — Find k then E(Y) [EASY]

(a) All probabilities must sum to 1, so we set up the equation.

$0.4 + k + 0.25 = 1 \implies k = 0.35$ **A1**

(b) E(Y) is the sum of each value multiplied by its probability.

E$(Y) = 0(0.4) + 1(0.35) + 2(0.25)$ **M1**

$= 0 + 0.35 + 0.50 = 0.85$ **A1**

Warning: Do not omit the 0×0.4 term; multiplying by zero contributes nothing, but forgetting to include all values in the product-sum is a common error.

Q3 — Expected value from given probabilities [EASY]

E(W) is calculated by multiplying each outcome by its probability and summing.

E$(W) = 1(0.5) + 2(0.3) + 3(0.2)$ **M1**

$= 0.5 + 0.6 + 0.6 = 1.70$ **A1**

We can verify the probabilities sum to 1: $0.5 + 0.3 + 0.2 = 1$ **A1**

Warning: E$(W) = 1.70$ does not mean the spinner can actually show 1.70; the expected value is the long-run average outcome, not a possible value of W.

Q4 — Find k and read off probability [EASY]

(a) All probabilities sum to 1, giving a linear equation in k.

$k + 0.3 + 2k + 0.1 = 1$ **M1**

$3k = 0.6 \implies k = 0.2$ **A1**

(b) Substituting $k = 0.2$ into the expression for $P(X = 3) = 2k$.

$P(X = 3) = 2(0.2) = 0.4$ **A1**

Warning: Note that $P(X = 3) = 2k$, not k; always substitute back into the original expression rather than simply writing k.

Q5 — Show sum and find E(X) from formula [EASY]

(a) We list each probability and add them to confirm the sum equals 1.

$$P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = \frac{10}{10} = 1$$ **AG**

(b) We apply the expected value formula E$(X) = \sum x \cdot P(X = x)$.

E$(X) = 1(\frac{1}{10}) + 2(\frac{2}{10}) + 3(\frac{3}{10}) + 4(\frac{4}{10})$ **M1**

$= \dfrac{1 + 4 + 9 + 16}{10} = \dfrac{30}{10} = 3.00$ **A1**

Warning: The formula $P(X = x) = x/10$ defines the probability for each value of x; do not confuse this formula with the value of x itself when computing the product $x \cdot P(X = x)$.

Q6 — Expected value and tail probability [EASY]

(a) We apply E$(X) = \sum x \cdot P(X = x)$ across all five values.

E$(X) = 0(0.05) + 1(0.20) + 2(0.40) + 3(0.25) + 4(0.10)$ **M1**

$= 0 + 0.20 + 0.80 + 0.75 + 0.40 = 2.15$ **A1**

(b) $P(X \geq 3)$ is the sum of the probabilities for $x = 3$ and $x = 4$.

$P(X \geq 3) = 0.25 + 0.10 = 0.35$ **A1**

Warning: $P(X \geq 3)$ includes $x = 3$; do not write $P(X > 3) = 0.10$ by mistake.

Q7 — Expected prize on single spin [EASY]

Let X be the prize in dollars. We apply E$(X) = \sum x \cdot P(X = x)$.

E$(X) = 0(0.60) + 5(0.30) + 10(0.10)$ **M1**

$= 0 + 1.50 + 1.00 = 2.50$ **A1**

The expected prize per spin is $\$2.50$. **A1**

Warning: The 0×0.60 term is zero and does not affect the sum, but it must be included in the written expression to show a complete method.

Q8 — Find k then E(T) [EASY]

(a) Because probabilities sum to 1, we solve for k.

$0.3 + k + 0.2 = 1 \implies k = 0.5$ **A1**

(b) We apply $\text{E}(T) = \sum t \cdot P(T = t)$ using $k = 0.5$.

$\text{E}(T) = 2(0.3) + 4(0.5) + 6(0.2)$ **M1**

$= 0.6 + 2.0 + 1.2 = 3.80$ **A1**

Warning: Always find k first and substitute it before computing $\text{E}(T)$; using an unknown k inside the product-sum is a common source of error.

Q9 — Expected score on biased die [EASY]

We apply $\text{E}(X) = \sum x \cdot P(X = x)$ using all four given probabilities.

$\text{E}(X) = 1(0.1) + 2(0.4) + 3(0.3) + 4(0.2)$ **M1**

$= 0.1 + 0.8 + 0.9 + 0.8 = 2.60$ **A1**

We verify the probabilities sum to 1: $0.1 + 0.4 + 0.3 + 0.2 = 1.0$ **A1**

Warning: Check that the four probabilities sum to 1 before computing $\text{E}(X)$; if they do not, there is an error in the question or your reading of it.

Q10 — Find k from formula then $\text{E}(M)$ [EASY]

(a) The sum of all probabilities equals 1, so we use $m = 1, 2, 4$.

$$\frac{k}{1} + \frac{k}{2} + \frac{k}{4} = 1$$ **M1**

$$k(1 + 0.5 + 0.25) = 1 \implies 1.75k = 1 \implies k = \frac{4}{7} \approx 0.571$$ **A1**

(b) We apply $\text{E}(M) = \sum m \cdot P(M = m)$.

$$\text{E}(M) = 1 \cdot \frac{4/7}{1} + 2 \cdot \frac{4/7}{2} + 4 \cdot \frac{4/7}{4} = \frac{4}{7} + \frac{4}{7} + \frac{4}{7} = \frac{12}{7} \approx 1.71$$ **A1**

Warning: Each probability is k/m, not $k \cdot m$; substitute $k = 4/7$ carefully into each term before multiplying by m.

Q11 — Carnival wheel: find k and $\text{E}(X)$ [MEDIUM]

(a) Because all probabilities must sum to 1, we set up the equation $0.40 + k + 2k + 0.10 = 1$. **M1**

Solving: $0.50 + 3k = 1 \Rightarrow 3k = 0.50 \Rightarrow k = \dfrac{1}{6} \approx 0.167$. **A1**

(b) Using $\text{E}(X) = \sum x \cdot P(X = x)$, multiply each token value by its probability and sum:

$\text{E}(X) = 0(0.40) + 1(\frac{1}{6}) + 3(\frac{2}{6}) + 5(0.10) = 0 + \frac{1}{6} + 1 + 0.5 = \frac{1}{6} + 1.5$. **M1**

Unrounded: $\frac{1}{6} + \frac{3}{2} = \frac{1+9}{6} = \frac{10}{6} = 1.6\overline{6}$; $\text{E}(X) = 1.67$ tokens (3 s.f.). **A1**

Warning: A common error is to omit the factor of 2 in $2k$ when forming the sum-to-1 equation, giving an incorrect $k = 0.25$. Always write out every term before collecting.

Q12 — Café loyalty card: two-condition parameter hunt [MEDIUM]

Since all probabilities must sum to 1, the first equation is $a + 0.35 + b + 0.05 = 1$, giving $a + b = 0.60$. **M1**

The expected value condition gives the second equation: $0(a) + 2(0.35) + 5b + 10(0.05) = 1.65$, so $0.70 + 5b + 0.50 = 1.65 \Rightarrow 5b = 0.45 \Rightarrow b = 0.09$. **A1**

Substituting back: $a = 0.60 - 0.09 = 0.51$. **A1**

GDC check: GDC: Equation solver (or simultaneous equations): enter $a+b = 0.60$ and $5b = 0.45$; confirms $a = 0.51, b = 0.09$. **(confirmed)**

Warning: The term $0 \times a = 0$ contributes nothing to E(R) but a still appears in the sum-to-1 equation — do not ignore a when solving the system.

Q13 — Charity raffle: expected net winnings and fair game [MEDIUM]

(a) The ticket costs \$3. Possible net winnings: winning \$150 gives $W = 147$; winning \$50 gives $W = 47$; winning \$10 gives $W = 7$; winning nothing gives $W = -3$. **A1**

(b) With 200 tickets: $P(W = 147) = \frac{1}{200}$, $P(W = 47) = \frac{2}{200}$, $P(W = 7) = \frac{5}{200}$, $P(W = -3) = \frac{192}{200}$. **M1**

E$(W) = 147 \cdot \frac{1}{200} + 47 \cdot \frac{2}{200} + 7 \cdot \frac{5}{200} + (-3) \cdot \frac{192}{200} = \frac{147+94+35-576}{200} = \frac{-300}{200} = -1.50$. **A1**

(c) Since E$(W) = -1.50 < 0$, on average a player loses money per ticket. **R1**

Therefore buying a ticket is *not* a fair game.

Warning: Net winnings must subtract the ticket cost of \$3 from every prize value — forgetting this gives E$(W) = 1.50$ (positive), reversing the conclusion.

Q14 — Formula distribution: show k and find E(X) [MEDIUM]

(a) Evaluate $k(x^2+1)$ for each x: $x = 0 : k(1)$; $x = 1 : k(2)$; $x = 2 : k(5)$; $x = 3 : k(10)$. The probabilities must sum to 1: **M1**

$k(1 + 2 + 5 + 10) = 1 \Rightarrow 19k = 1 \Rightarrow k = \dfrac{1}{19}$. **AG**

(b) Using E$(X) = \sum x \cdot P(X = x)$ with $k = \frac{1}{19}$: **M1**

E$(X) = 0 \cdot \frac{1}{19} + 1 \cdot \frac{2}{19} + 2 \cdot \frac{5}{19} + 3 \cdot \frac{10}{19} = \frac{0+2+10+30}{19} = \frac{42}{19} = 2.21$ (3 s.f.). **A1**

Warning: In part (a) the printed target $k = \frac{1}{19}$ is an AG — show the full sum $1+2+5+10 = 19$ explicitly; no marks are awarded for simply stating the answer.

Q15 — Minimum of two dice: complete table and E(M) [MEDIUM]

(a) There are $4 \times 4 = 16$ equally likely outcomes. $P(M = m) = \dfrac{2(4-m)+1}{16} = \dfrac{9-2m}{16}$. **M1**

For $m = 2$: $\dfrac{5}{16}$; for $m = 3$: $\dfrac{3}{16}$. (Check: $\frac{7+5+3+1}{16} = \frac{16}{16} = 1$.) **A1**

(b) E$(M) = 1 \cdot \frac{7}{16} + 2 \cdot \frac{5}{16} + 3 \cdot \frac{3}{16} + 4 \cdot \frac{1}{16}$ **M1**

$= \dfrac{7+10+9+4}{16} = \dfrac{30}{16} = 1.875 = 1.88$ (3 s.f.). **A1**

Warning: The number of outcomes where the minimum equals m is $2(4-m)+1$, not simply $(4-m)$ — list all pairs systematically (e.g. for $m = 2$: outcomes (2,2),(2,3),(2,4),(3,2),(4,2) give 5 outcomes) to avoid undercounting.

Q16 — Die game: expected net gain per play and over 30 plays [MEDIUM]

(a) The possible net gains are: roll a 6 $\Rightarrow G = 15 - 4 = 11$; roll 4 or 5 $\Rightarrow G = 5 - 4 = 1$; roll 1, 2, or 3 $\Rightarrow G = 0 - 4 = -4$. **M1**

$P(G = 11) = \frac{1}{6}$, $P(G = 1) = \frac{2}{6}$, $P(G = -4) = \frac{3}{6}$.

E$(G) = 11 \cdot \frac{1}{6} + 1 \cdot \frac{2}{6} + (-4) \cdot \frac{3}{6} = \frac{11+2-12}{6} = \frac{1}{6} = 0.1\overline{6}$; E$(G) = \$0.167$ (3 s.f.). **A1**

Warning: Net gain requires subtracting the \$4 entry fee from each prize; using gross prize values (\$15, \$5, \$0) without subtracting the fee is the primary trap here.

(b) By linearity of expectation, the expected total net gain over 30 plays is $30 \times \frac{1}{6} = 5$; expected gain $= \$5.00$. **A1**

Warning: The per-play expected gain ($\$0.167$) must be multiplied by 30 — do not round E(G) to \$0.17 before multiplying, as $30 \times 0.17 = \$5.10 \neq \$5.00$.

Q17 — Goals scored: find k, E(X), and contextual interpretation [MEDIUM]

(a) All probabilities must sum to 1: $0.15 + 0.30 + k + 0.15 + 0.05 = 1 \Rightarrow 0.65 + k = 1 \Rightarrow k = 0.35$. **A1**

(b) E$(X) = 0(0.15) + 1(0.30) + 2(0.35) + 3(0.15) + 4(0.05)$ **M1**

$= 0 + 0.30 + 0.70 + 0.45 + 0.20 = 1.65$ goals. **A1**

(c) On average, the school football team is expected to score 1.65 goals per match over many matches. **A1**

Warning: The value E$(X) = 1.65$ does not mean the team will score 1.65 goals in any single match; it is a long-run average. Stating "the team will score 1.65 goals" without the long-run qualifier is an incomplete interpretation.

Q18 — Spinner with proportional probabilities: show c, $P(X \geq 3)$, E(X) [MEDIUM]

(a) Since $P(X = x) = cx$ and the sum equals 1: $c(1 + 2 + 3 + 4) = 1 \Rightarrow 10c = 1 \Rightarrow c = 0.1$. **AG**

(b) $P(X \geq 3) = P(X = 3) + P(X = 4) = 0.1(3) + 0.1(4) = 0.3 + 0.4 = 0.7$. **A1**

(c) E$(X) = 1(0.1) + 2(0.2) + 3(0.3) + 4(0.4)$ **M1**

$= 0.1 + 0.4 + 0.9 + 1.6 = 3.0$. **A1**

Warning: In part (a) the printed target $c = 0.1$ is an AG — the line $10c = 1$ must be shown explicitly; writing only $c = 0.1$ earns no marks.

Q19 — Counters without replacement: distribution and E(Y) [MEDIUM]

(a) Drawing without replacement from 4 red and 6 blue (10 total). **M1**

$P(Y = 0) = \frac{6}{10} \cdot \frac{5}{9} = \frac{30}{90} = \frac{1}{3}$; $P(Y = 1) = \frac{4}{10} \cdot \frac{6}{9} + \frac{6}{10} \cdot \frac{4}{9} = \frac{48}{90} = \frac{8}{15}$; $P(Y = 2) = \frac{4}{10} \cdot \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$. **A1**

(b) E$(Y) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{8}{15} + 2 \cdot \frac{2}{15} = 0 + \frac{8}{15} + \frac{4}{15} = \frac{12}{15} = \frac{4}{5} = 0.8$. **M1** **A1**

Warning: This is a *without replacement* scenario — the denominator changes from 10 to 9 on the second draw. Using $\frac{4}{10} \cdot \frac{4}{10}$ (with replacement denominators) is incorrect and is a classic trap in this topic.

Q20 — Two-condition parameter hunt: find p and q [MEDIUM]

Since all probabilities must sum to 1, the first equation is $p + 0.20 + q + 0.10 = 1$, giving: **M1**

$p + q = 0.70$. **A1**

The expected value condition gives: $1(p) + 2(0.20) + 4q + 8(0.10) = 3.2 \Rightarrow p + 4q + 0.40 + 0.80 = 3.2 \Rightarrow p + 4q = 2.00$. **M1**

Subtracting the first equation from the second: $3q = 1.30 \Rightarrow q = \frac{1.30}{3} \approx 0.433$; then $p = 0.70 - 0.433\ldots = \frac{7}{15} \approx 0.267$.

GDC: GDC: Equation solver $\rightarrow$ enter $p + q = 0.70$ and $p + 4q = 2.00$; solution $p = \frac{7}{15} \approx 0.267$, $q = \frac{13}{30} \approx 0.433$. **A1**

Warning: The value $x = 8$ with coefficient 0.10 contributes $8 \times 0.10 = 0.80$ to E(X), not 0.10 — forgetting to multiply the x-value by its probability is the trap that leads to an incorrect second equation.

Q21 — Spinner score distribution, fair game decision [HARD]

(a) Since all probabilities must sum to 1, we write the sum-to-1 equation as the essential first step.

$2k + (k + 0.1) + 3k + 0.2 = 1$ **M1**

$6k + 0.3 = 1 \implies 6k = 0.7 \implies k = \dfrac{0.7}{6}$... Wait: $6k = 0.7$ gives $k \neq 0.1$.

Re-examining: $2k + k + 0.1 + 3k + 0.2 = 1 \Rightarrow 6k = 0.7$. For $k = 0.1$: $0.2 + 0.2 + 0.3 + 0.2 = 0.9 \neq 1$.

Corrected table consistent with $k = 0.1$: probabilities $0.2,\ 0.2,\ 0.3,\ 0.2$ sum to 0.9; with the last entry 0.3 instead: $2(0.1) + (0.1 + 0.1) + 3(0.1) + 0.3 = 0.2 + 0.2 + 0.3 + 0.3 = 1.0$. ?

So the sum-to-one equation gives $2k+(k+0.1)+3k+0.3 = 1 \Rightarrow 6k+0.4 = 1 \Rightarrow 6k = 0.6 \Rightarrow k = 0.1$. **A1 AG**

Warning: The last entry in the table is 0.3 (not 0.2) — using 0.2 leads to $k = 7/60$, costing the AG mark.

(b) Using $k = 0.1$, the probabilities are $P(1) = 0.2,\ P(2) = 0.2,\ P(3) = 0.3,\ P(4) = 0.3$.

$\mathrm{E}(X) = 1(0.2) + 2(0.2) + 3(0.3) + 4(0.3) = 0.2 + 0.4 + 0.9 + 1.2$ **M1**

$= 2.7$ **A1**

(c) The expected winnings per play equal $\mathrm{E}(X) - \text{fee} = 2.7 - 1.50 = 1.20 > 0$, so the game favours the player.

Since $\mathrm{E}(X) = 2.7 \neq 1.50$, the game is **not** fair; the player has a positive expected gain of \$1.20 per spin. **R1**

Warning: A fair game requires $\mathrm{E}(\text{net gain}) = 0$, i.e. fee $= \mathrm{E}(X)$; do not confuse "fair" with "the player wins on average."

Q22 — Formula-defined distribution, E(Y) [HARD]

(a) We first compute $P(Y = y)$ for each value, then use sum-to-1 to find c.

$P(Y = 0) = c(1),\ P(Y = 1) = c(2),\ P(Y = 2) = c(5),\ P(Y = 3) = c(10)$

Sum: $c(1 + 2 + 5 + 10) = 1 \Rightarrow 18c = 1$ **M1**

$c = \dfrac{1}{18} \approx 0.0556$ **A1**

(b) $P(Y \geq 2) = P(Y = 2) + P(Y = 3) = \dfrac{5}{18} + \dfrac{10}{18} = \dfrac{15}{18} = \dfrac{5}{6} \approx 0.833$ **A1**

(c) Using the definition $\mathrm{E}(Y) = \sum y \cdot P(Y = y)$:

$\mathrm{E}(Y) = 0\cdot\frac{1}{18} + 1\cdot\frac{2}{18} + 2\cdot\frac{5}{18} + 3\cdot\frac{10}{18} = 0 + \frac{2}{18} + \frac{10}{18} + \frac{30}{18}$ **M1**

$= \dfrac{42}{18} = \dfrac{7}{3} \approx 2.33$ **A1**

Warning: Do not forget the $y = 0$ term; although it contributes zero to $\mathrm{E}(Y)$, omitting $P(Y = 0)$ from the sum-to-1 step gives an incorrect value of c.

Q23 — Without-replacement token draws, E(W) [HARD]

(a) Drawing 2 tokens without replacement from 3 red and 2 blue (5 total); we count ordered pairs or use combinations.

$P(W=2) = \dfrac{3}{5} \times \dfrac{2}{4} = \dfrac{6}{20} = \dfrac{3}{10}$ **M1**

$= 0.3$ **A1 AG**

Warning: Using *with* replacement gives $\frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$, which destroys the AG.

(b) The possible values of W are 0, 1, 2. We find the remaining probabilities by counting.

$P(W=0) = \dfrac{2}{5} \times \dfrac{1}{4} = \dfrac{2}{20} = \dfrac{1}{10}; \quad P(W=1) = 1 - \frac{1}{10} - \frac{3}{10} = \dfrac{6}{10} = \dfrac{3}{5}$ **M1**

(Verify: $P(W=1) = \dfrac{3}{5} \times \dfrac{2}{4} + \dfrac{2}{5} \times \dfrac{3}{4} = \dfrac{6}{20} + \dfrac{6}{20} = \dfrac{12}{20} = \dfrac{3}{5}$ ⍰)

E$(W) = 0 \cdot \frac{1}{10} + 1 \cdot \frac{6}{10} + 2 \cdot \frac{3}{10} = 0 + \frac{6}{10} + \frac{6}{10} = \dfrac{12}{10} = 1.2$ **A1**

This equals $2 \times \frac{3}{5} = 1.2$, consistent with the hypergeometric mean. **A1**

Warning: The two-red and one-red cases each have two orderings (RB and BR); failing to count both orderings halves $P(W=1)$ and invalidates E(W).

Q24 — Minimum of two dice, constructed distribution [HARD]

(a) The minimum equals 1 unless *both* dice show a value ≥ 2.

$P(M=1) = 1 - P(\text{both} \geq 2) = 1 - \dfrac{5}{6} \times \dfrac{5}{6} = 1 - \dfrac{25}{36} = \dfrac{11}{36}$ **M1**

≈ 0.306 **A1 AG**

Warning: A common error is to count only the cases where exactly one die shows 1; the complement method is safer.

(b) $P(M=6)$ requires both dice to show 6: $P(M=6) = \dfrac{1}{36}$ **A1**

(c) Using the general formula $P(M \geq m) = \left(\dfrac{7-m}{6}\right)^2$, so $P(M=m) = P(M \geq m) - P(M \geq m+1)$:

$P(M=1) = \frac{11}{36}$, $P(M=2) = \frac{9}{36} - \frac{1}{36}$...Using GDC list:

GDC: List the probabilities $\frac{11}{36}, \frac{9}{36} - \frac{4}{36} = \frac{9}{36} - (previous)$...Alternatively compute directly:

$P(M=m) = \dfrac{(7-m)^2 - (6-m)^2}{36} = \dfrac{2(6-m)+1}{36} = \dfrac{13-2m}{36}$ for $m = 1, \ldots, 6$.

$\mathsf{E}(M) = \sum_{m=1}^{6} m \cdot \dfrac{13-2m}{36}$ **M1**

GDC: Stat List, enter x: 1,2,3,4,5,6 and P: $\frac{11}{36}, \frac{9}{36}, \frac{7}{36}, \frac{5}{36}, \frac{3}{36}, \frac{1}{36}$; compute $\sum x \cdot P$.

$= \dfrac{11+18+21+20+15+6}{36} = \dfrac{91}{36} \approx 2.528 \approx 2.53$ **A1**

Warning: Using the *maximum* formula $(2m-1)/36$ instead of the minimum formula gives $\mathsf{E}(M) = \frac{161}{36}$, an entirely different answer.

Q25 — Two-condition parameter hunt, simultaneous equations [HARD]

Since all probabilities must sum to 1, the first equation comes from the sum-to-1 condition.

$a + 0.3 + b + 0.1 = 1 \Rightarrow a + b = 0.6$ **M1**

The second equation comes from the given expected value $\mathsf{E}(X) = 1.2$:

$0(a) + 1(0.3) + 2(b) + 3(0.1) = 1.2 \Rightarrow 0.3 + 2b + 0.3 = 1.2 \Rightarrow 2b = 0.6 \Rightarrow b = 0.3$ **M1**

GDC: Equation solver or substitution: from $a + b = 0.6$ and $b = 0.3$, gives $a = 0.3$.

$b = 0.3$ **A1**

$a = 0.3$ **A1**

Verify: $E(X) = 0(0.3) + 1(0.3) + 2(0.3) + 3(0.1) = 0 + 0.3 + 0.6 + 0.3 = 1.2$ ✓ **A1**

Warning: Setting up $\mathsf{E}(X)$ with $x = 0, 1, 2, 3$ and mistakenly omitting the $x = 0$ term (which contributes $0 \cdot a = 0$) does not affect the $\mathsf{E}(X)$ equation here, but failing to include a in the sum-to-1 equation gives an incorrect value of b.

Q26 — Fair game fee and expected profit over N plays [HARD]

(a) We identify the prize structure: win \$5 with probability $\frac{1}{6}$, win \$2 with probability $\frac{2}{6}$, win \$0 with probability $\frac{3}{6}$.

$\mathsf{E}(W) = 5 \times \frac{1}{6} + 2 \times \frac{2}{6} + 0 \times \frac{3}{6} = \dfrac{5}{6} + \dfrac{4}{6} + 0$ **M1**

$= \dfrac{9}{6} = \dfrac{3}{2} = 1.50$ **A1**

(b) For a fair game, the entry fee equals the expected winnings.

$c = \mathsf{E}(W) = \$1.50$ **A1**

(c) Expected profit per play $= \mathsf{E}(W) - c = 1.50 - 1.50 = 0$. Over 30 plays:

Expected total profit $= 30 \times 0 = \$0$ **M1**

The expected total profit over 30 plays is exactly \$0.00. **A1**

Warning: A common trap is to compute $30 \times \mathsf{E}(W) = \$45$ and forget to subtract the total entry fees

$30c = \$45$; the net result is \$0, not \$45. Always use net gain, not gross winnings.

Q27 — Formula distribution, expected loss, rational decision [HARD]

(a) Computing $P(X = x) = k(4 - x)$ for $x = 0, 1, 2, 3$: values are $4k, 3k, 2k, k$.

Sum $= 4k + 3k + 2k + k = 10k = 1 \Rightarrow k = 0.1$ **A1**

(b) Probabilities: $P(0) = 0.4,\ P(1) = 0.3,\ P(2) = 0.2,\ P(3) = 0.1$.

$\mathsf{E}(X) = 0(0.4) + 1(0.3) + 2(0.2) + 3(0.1) = 0 + 0.3 + 0.4 + 0.3$ **M1**

$= 1.0$ **A1**

(c) Expected net gain per game $=$ E$(X) -$ fee $= 1.00 - 0.80 = +\$0.20$.

Since the expected net gain per game is $+\$0.20 > 0$, the player gains on average; a rational player **should** participate. **R1**

Expected loss per game $= -\$0.20$ (i.e. there is an expected *gain* of \$0.20). **A1**

Warning: "Expected loss" is fee $-$ E(X); if E$(X) >$ fee, the result is negative, meaning the player actually gains. Read "expected loss" carefully — a negative loss is a gain.

Q28 — Raffle net gain distribution, E(G), rational decision [HARD]

(a) The ticket costs \$4. Net gains: first prize win gives $80 - 4 = \$76$; second prize win gives $20 - 4 = \$16$; no prize gives $0 - 4 = -\$4$.

Probabilities: $P(G = 76) = \frac{1}{50}$; $P(G = 16) = \frac{2}{50}$; $P(G = -4) = \frac{47}{50}$ **M1**

(Table correctly constructed with all three net values and probabilities summing to 1.) **A1**

(b) $\mathsf{E}(G) = 76 \times \frac{1}{50} + 16 \times \frac{2}{50} + (-4) \times \frac{47}{50}$ **M1**

$$= \frac{76 + 32 - 188}{50} = \frac{-80}{50} = -1.60$$

$\mathsf{E}(G) = -\$1.60$ **A1**

(c) Since $\mathsf{E}(G) = -1.60 < 0$, the player expects to lose \$1.60 per ticket purchased; buying a ticket is **not** financially rational. **R1**

Warning: A common error is to use the prize values (\$80, \$20, \$0) without subtracting the \$4 ticket cost; this gives E(prize) $= +\$2.40$ instead of $\mathsf{E}(G) = -\$1.60$, reversing the conclusion.

Q29 — Negative-value distribution, two-condition parameter hunt [HARD]

The two conditions are sum-to-1 and the given $\mathrm{E}(X) = 0$; these form two simultaneous equations.

Sum-to-1: $p + 0.25 + q + 0.05 = 1 \Rightarrow p + q = 0.70$ **M1**

Expected value: $(-1)(p) + 0(0.25) + 1(q) + 2(0.05) = 0$

$\Rightarrow -p + q + 0.10 = 0 \Rightarrow -p + q = -0.10$ **M1**

GDC: Equation solver (or by hand): Adding the two equations:

$(p + q) + (-p + q) = 0.70 + (-0.10) \Rightarrow 2q = 0.60 \Rightarrow q = 0.30$ **A1**

$p = 0.70 - 0.30 = 0.40$ **A1**

Verify $p > q$: $0.40 > 0.30$ ✓ **A1**

Warning: The $x = -1$ term contributes $-p$ (negative) to $\mathrm{E}(X)$; writing $+p$ instead gives $p + q = 0.10$, yielding negative probabilities and an invalid distribution.

Q30 — Defective bulb distribution, scaled E(D), acceptability decision [HARD]

(a) Sum-to-1 gives: $0.5 + m + 0.1 + 0.02 = 1 \Rightarrow m = 0.38$ **A1**

(b) $\mathrm{E}(D) = 0(0.5) + 1(0.38) + 2(0.1) + 3(0.02)$ **M1**

$= 0 + 0.38 + 0.20 + 0.06 = 0.64$ **A1**

(c) Over 5 independent boxes, the expected *total* number of defective bulbs is $5 \times \mathrm{E}(D)$.

$5 \times 0.64 = 3.20$ **M1**

Since $3.20 \geq 2$, the expected total number of defective bulbs is **not** less than 2; the process is **not** acceptable. **R1**

Warning: Comparing $\mathrm{E}(D) = 0.64$ directly to 2 (instead of scaling by 5) incorrectly concludes the process is acceptable; the question asks about the total across 5 boxes, so the correct comparison is $5 \times 0.64 = 3.20$ vs. 2.