IB Math AI SL — Topic 5

Differentiation

Derivatives & Powers of x · Tangents & Normals · Stationary Points · Optimisation Models

Difficulty	**EASY**	**MEDIUM**	**HARD**
Questions	10	10	10
Total Marks	30	40	50

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Name:______________________________ Date:_______________

Score:__________/120

Instructions: Answer all questions. Show all working. Give answers correct to 3 significant figures unless stated otherwise. A graphic display calculator is required throughout; support GDC answers with the values entered and the feature used.

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Key Formulae

- **Power Rule:** $\dfrac{d}{dx}(x^n) = nx^{n-1}$ (valid for all $n \in \mathbb{R}$)
- **Constant and Linear Rules:** $\dfrac{d}{dx}(c) = 0$; $\dfrac{d}{dx}(ax + b) = a$
- **Rewriting before differentiating:** $\dfrac{a}{x^n} = ax^{-n}$ ⍰ $\dfrac{d}{dx}(ax^{-n}) = -nax^{-n-1}$
- **Gradient of tangent at** $x = a$**:** $m_{\text{tan}} = f'(a)$; equation: $y - f(a) = f'(a)(x - a)$
- **Gradient of normal at** $x = a$**:** $m_{\text{norm}} = -\dfrac{1}{f'(a)}$; equation: $y - f(a) = -\dfrac{1}{f'(a)}(x - a)$
- **Increasing / Decreasing:** $f'(x) > 0 \Rightarrow$ increasing; $f'(x) < 0 \Rightarrow$ decreasing
- **Stationary points:** solve $f'(x) = 0$; classify by sign of f' either side (sign table or GDC graph)

GDC Tips

- **Evaluating the derivative at a point:** Enter $f(x)$ in `Graph`, then use `Analysis` $\to$ `Sketch` $\to$ `Tangent` at the desired x-value; the gradient is displayed, or use `Calc` $\to$ `dy/dx` and type the x-value.
- **Finding stationary points (solve $f'(x) = 0$):** Enter $f'(x)$ in `Equation/Graph`, then `Analysis` $\to$ `G-Solve` $\to$ `Root` to read each zero of the derivative directly.
- **Classifying a stationary point:** Graph $f(x)$, zoom around the stationary point, then use `G-Solve` $\to$ `Maximum` or `G-Solve` $\to$ `Minimum` to confirm the type and read the coordinates.
- **Tangent/Normal line equations:** In `Graph`, enter $f(x)$, go to `Analysis` $\to$ `Sketch` $\to$ `Tangent` (or `Normal`), input the x-value; the GDC draws the line and shows its equation.
- **Finding the x-value where gradient equals a given value k:** Enter $Y_1 = f'(x)$ and $Y_2 = k$, then `Analysis` $\to$ `G-Solve` $\to$ `Intersect` to find x.
- **Checking increasing/decreasing intervals:** Graph $f'(x)$ and inspect where it lies above ($f' > 0$) or below ($f' < 0$) the x-axis using `G-Solve` $\to$ `Root` for boundary points.

Common Mistakes to Avoid

1. **Sign error on negative powers.** When differentiating $\frac{a}{x^2} = ax^{-2}$, the power rule gives $-2ax^{-3}$, *not* $+2ax^{-3}$. Always rewrite the term explicitly as ax^{-n} before applying the rule, and double-check the sign of $n \cdot a$ when n is negative.
2. **Forgetting to rewrite before differentiating.** You cannot apply the power rule to $\frac{3}{x^2}$ in fraction form. Rewrite as $3x^{-2}$ first — this step must be shown in your working or method marks are lost.
3. **Using the tangent gradient for the normal.** The normal at a point is *perpendicular* to the tangent, so its gradient is $-\frac{1}{f'(a)}$. Failing to take the negative reciprocal and instead using $f'(a)$ directly gives an equation for the tangent, not the normal.
4. **Asserting a maximum or minimum without evidence.** Stating "it is a minimum" earns no marks. You must provide a sign table of f' on both sides of the stationary point, or describe the GDC graph showing the function changes from decreasing to increasing (or vice versa).
5. **Premature rounding in chained calculations.** If you round $f'(a)$ to 3 s.f. before using it in the tangent or normal equation, the final equation will be inaccurate and follow-through marks may not recover the loss. Keep unrounded values until the last step, then round.
6. **Misidentifying increasing/decreasing intervals from $f'(x) = 0$ alone.** The boundary of an increasing interval is where $f'(x) = 0$, but you must check the *sign* of f' on each side. A stationary point that is a point of inflection does not change the increasing/decreasing status, so listing it as a boundary of an interval is incorrect.

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Section A — Easy

EASY

1. The height of water in a fountain, h metres, is modelled by

$$f(x) = 3x^2 - 4x + 1,$$

where x is the horizontal distance in metres from the base.

(a) Write down the value of $f(0)$. **[1 marks]**

(b) Find $f'(x)$. **[2 marks]**

2. A jogger's distance from a starting point, d metres, is modelled by

$$g(t) = 5t^2 + 2t,$$

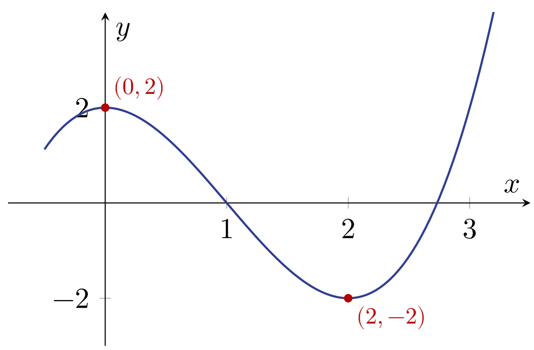
where t is the time in seconds.
Find $g'(t)$ and hence find the rate of change of distance at $t = 3$ seconds. **[3 marks]**

3. The temperature of a cooling drink, T degrees Celsius, is given by

$$T(x) = 20 - 6x + x^2,$$

where x is the time in minutes after being removed from a fridge.
Find the gradient of the curve at $x = 4$. **[3 marks]**

4. A curve has equation $y = x^3 - 3x^2 + 2$. The diagram below shows the curve with its stationary points labelled.



(a) Find $\dfrac{dy}{dx}$. **[2 marks]**

(b) Write down the x-coordinates of the stationary points. **[1 marks]**

5. The profit, P thousand dollars, made by a small bakery is modelled by

$$P(n) = -2n^2 + 12n - 10,$$

where n is the number of batches of bread baked per day.
Find the value of n that maximises the profit. **[3 marks]**

6. A curve is defined by $y = 4x^3 - x$. Find the equation of the tangent to the curve at the point where $x = 1$. **[3 marks]**

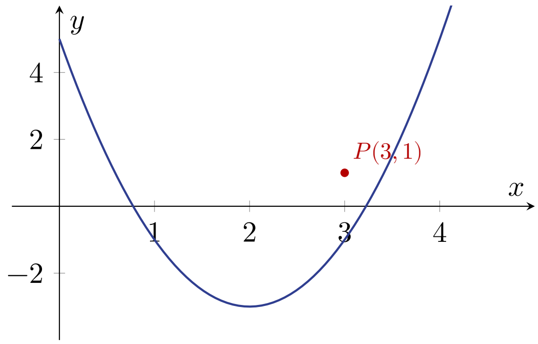
7. The depth of water in a harbour, D metres, at time t hours after midnight is modelled by

$$D(t) = t^2 - 8t + 20, \quad 0 \leq t \leq 8.$$

(a) Find $D'(t)$. **[1 marks]**

(b) Find the interval of t for which the depth is decreasing. **[2 marks]**

8. The curve $y = 2x^2 - 8x + 5$ is shown below, with point P labelled.



Find the equation of the normal to the curve at $P(3, 1)$. **[3 marks]**

9. A function is given by $f(x) = x^3 - 6x^2 + 9x + 1$.

(a) Find $f'(x)$. **[1 marks]**

(b) Determine the interval on which f is increasing. **[2 marks]**

10. A particle moves along a straight track. Its displacement, s metres, from a fixed point is given by

$$s(t) = 3t^2 - 6t + 4,$$

where t is the time in seconds, $t \geq 0$.
Find the time at which the particle is momentarily at rest (i.e. has zero velocity). **[3 marks]**

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Section B — Medium

MEDIUM

11. The depth of water in a harbour, d metres, at time t hours after midnight is modelled by

$$d(t) = 0.5t^3 - 6t^2 + 18t + 2, \quad 0 \leq t \leq 8.$$

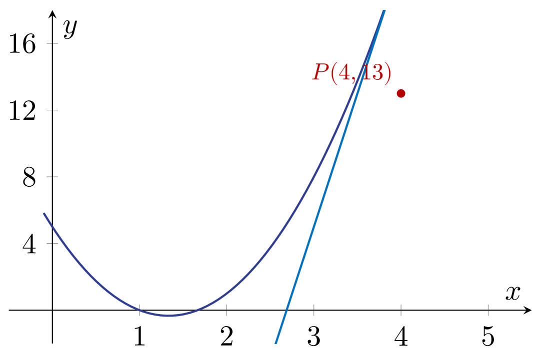
(a) Find $d'(t)$. **[2 marks]**

(b) Find the rate of change of depth at $t = 3$. State the units of your answer. **[2 marks]**

12. A curve has equation $f(x) = 3x^2 - 8x + 5$.

(a) Find the gradient of the curve at the point where $x = 4$. **[2 marks]**

(b) Find the equation of the tangent to the curve at $x = 4$. Give your answer in the form $y = mx + c$. **[2 marks]**



13. A function is defined by $f(x) = x^3 - 6x^2 + 9x + 1$.

(a) Find the values of x at which f has stationary points. **[2 marks]**

(b) Determine whether each stationary point is a local maximum or a local minimum. Justify your answer. **[2 marks]**

14. The profit, P thousand dollars, made by a small business selling x hundred units of a product is modelled by

$$P(x) = -2x^2 + 12x - 7, \quad 0 \leq x \leq 6.$$

(a) Find the value of x that maximises the profit. **[2 marks]**

(b) Find the maximum profit. **[2 marks]**

15. A curve is given by $g(x) = \dfrac{4}{x^2} + 3x,\ x \neq 0.$

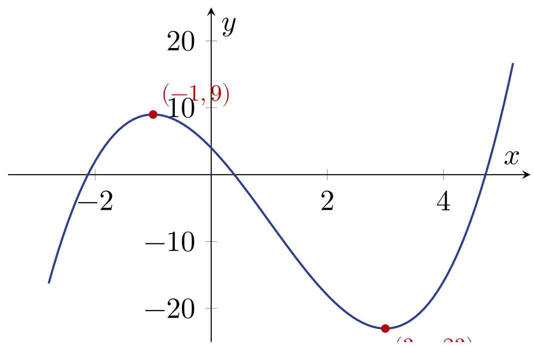
(a) Rewrite $g(x)$ using a negative power of x and hence find $g'(x)$. **[2 marks]**

(b) Find the gradient of g at $x = 2$. **[2 marks]**

16. A function is given by $h(x) = x^3 - 3x^2 - 9x + 4$.

(a) Find the interval on which h is decreasing. **[2 marks]**

(b) Find the coordinates of the local minimum of h. **[2 marks]**



17. The curve C has equation $f(x) = 2x^2 - 5x + 1$.

(a) Find the equation of the normal to C at the point where $x = 3$. **[3 marks]**

(b) Write down the gradient of the tangent to C at $x = 3$. **[1 marks]**

18. The height, h metres, of a ball t seconds after it is launched is given by

$$h(t) = -4.9t^2 + 14.7t + 1.2, \quad t \geq 0.$$

(a) Find $h'(t)$. **[1 marks]**

(b) Find the time at which the ball reaches its maximum height. **[2 marks]**

(c) State what $h'(t) < 0$ means in this context. **[1 marks]**

19. A curve has equation $f(x) = x^3 - 12x + 2$.

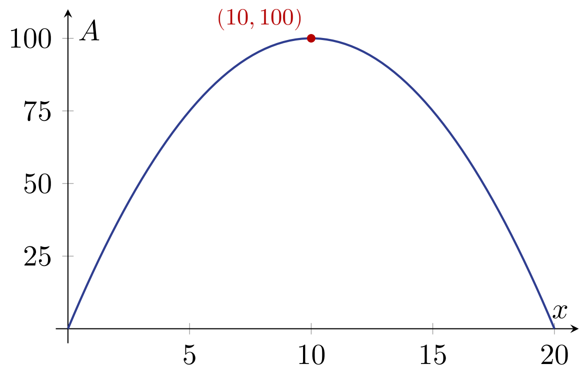
(a) Find the x-coordinates of the points on the curve where the gradient equals 0. **[1 marks]**

(b) Find the x-coordinates of the points on the curve where the gradient equals 24. **[3 marks]**

20. A rectangular garden has a fixed perimeter of 40 m. Let x metres be the width of the garden.

(a) Show that the area of the garden can be written as $A(x) = 20x - x^2$. **[1 marks]**

(b) Find the value of x that gives the maximum area, and state the maximum area. **[3 marks]**



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Section C — Hard

HARD

21. The profit, P (thousands of dollars), earned by a market stall selling handmade candles depends on the price per candle, x (dollars), where $2 \leq x \leq 20$. The profit function is given by

$$P(x) = -2x^3 + 30x^2 - 100x + 80.$$

(a) Find $P'(x)$. **[1 marks]**

(b) Find the value of x that maximises the profit. Justify your answer. **[3 marks]**

(c) State the maximum profit, giving your answer in dollars. **[1 marks]**

22. A function is defined as $f(x) = 3x^2 - \dfrac{48}{x^2}$ for $x > 0$.

(a) Show that $f'(x) = 6x + \dfrac{96}{x^3}$. **[2 marks]**

(b) Determine whether f is increasing or decreasing for all $x > 0$. Justify your answer. **[2 marks]**

(c) Find the equation of the tangent to f at the point where $x = 2$. **[1 marks]**

23. A rectangular swimming pool is to be built with one side along a garden wall. Fencing is needed only for the three remaining sides. The total length of fencing available is 60 metres. Let x metres be the length of each of the two equal sides perpendicular to the wall, and let A m^2 be the area of the pool.

(a) Show that $A = 60x - 2x^2$. **[1 marks]**

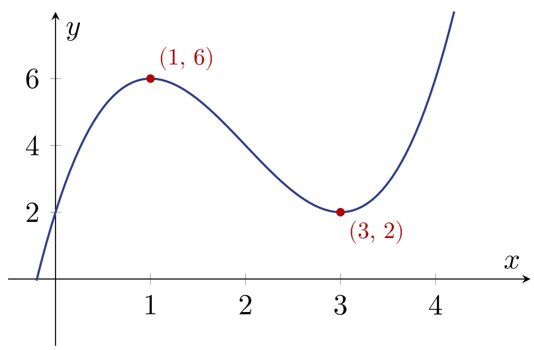
(b) Find $\dfrac{dA}{dx}$. **[1 marks]**

(c) Find the value of x that gives the maximum area. **[1 marks]**

(d) Hence find the maximum area of the pool. **[1 marks]**

(e) Explain why the model $A = 60x - 2x^2$ is not appropriate when $x = 35$. **[1 marks]**

24. Let $g(x) = x^3 - 6x^2 + 9x + 2$.



(a) Write down the coordinates of the local maximum and local minimum of g. **[1 marks]**

(b) Find the equation of the normal to g at the point where $x = 2$. **[3 marks]**

(c) Find the values of x for which g is increasing. **[1 marks]**

25. The temperature of a metal rod, T °C, is modelled by

$$T(t) = \frac{120}{t} + 4t - 10, \quad t > 0,$$

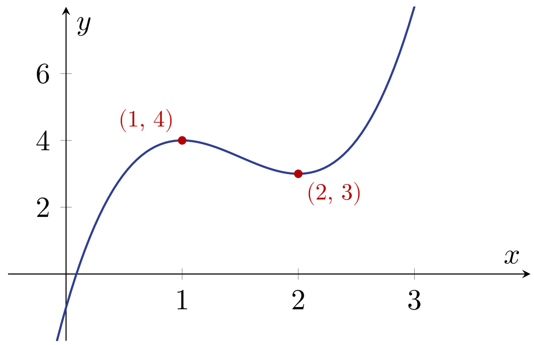
where t is the time in minutes after heating begins.

(a) Rewrite $T(t)$ using a negative power of t before differentiating. Hence find $T'(t)$. **[2 marks]**

(b) Find the time t at which the temperature is at a minimum. Justify your answer. **[2 marks]**

(c) Find the minimum temperature of the rod. **[1 marks]**

26. Let $h(x) = 2x^3 - 9x^2 + 12x - 1$.



(a) Find $h'(x)$. **[1 marks]**

(b) Find the gradient of h at $x = 3$. **[1 marks]**

(c) Find the equation of the tangent to h at $x = 3$. **[2 marks]**

(d) Determine the value(s) of x where the gradient of h equals 0. **[1 marks]**

27. A drone flies along a straight path. Its height, H metres, above the ground is modelled by

$$H(t) = -t^3 + 9t^2 - 15t + 30, \quad 0 \leq t \leq 8,$$

where t is the time in seconds after the drone is launched.

(a) Find $H'(t)$. **[1 marks]**

(b) Find the times at which the drone is at a stationary point in its flight. **[1 marks]**

(c) Determine which stationary point is a local maximum and which is a local minimum. Justify your answer using a sign diagram of $H'(t)$. **[2 marks]**

(d) Find the maximum height of the drone in the interval $0 \leq t \leq 8$. **[1 marks]**

28. A function is given by $f(x) = ax^2 + bx + 3$, where a and b are constants.
It is known that $f'(1) = 0$ and $f(1) = 7$.

(a) Show that $a + b = 0$ and $a + b = 4$. Identify the error in this approach and instead form two correct equations in a and b. **[2 marks]**

(b) Solve the two correct equations to find a and b. **[2 marks]**

(c) Write down the coordinates of the stationary point of f. **[1 marks]**

29. The curve C has equation $y = x^3 - 3x^2 - 9x + 5$.

(a) Find $\dfrac{dy}{dx}$. **[1 marks]**

(b) Find the coordinates of the two stationary points on C. **[2 marks]**

(c) The point $P(-1, 10)$ lies on C. Find the equation of the normal to C at P. **[2 marks]**

30. A container is made in the shape of a closed cylinder. The total surface area of the cylinder is 300π cm^2. The radius of the cylinder is r cm and the height is h cm.

(a) Show that the volume of the cylinder can be written as

$$V = 150\pi r - \pi r^3.$$

[2 marks]

(b) Find $\dfrac{dV}{dr}$. **[1 marks]**

(c) Find the value of r that maximises the volume. Justify your answer. **[2 marks]**

Worked Solutions

Q1 — Differentiating a quadratic height model [EASY]

(a) Substitute $x = 0$ directly into $f(x)$ to evaluate the initial height.

$$f(0) = 3(0)^2 - 4(0) + 1 = 1$$

A1

(b) Differentiate each term using the power rule, reducing each power of x by one and multiplying by the original power.

$$f'(x) = 6x - 4$$

A1A1

Warning: The constant term $+1$ vanishes on differentiation; a common error is to write $f'(x) = 6x - 4 + 1$.

Q2 — Rate of change of distance [EASY]

Differentiate $g(t) = 5t^2 + 2t$ term by term using the power rule to find the rate of change of distance with respect to time.

$$g'(t) = 10t + 2$$

M1A1

Substitute $t = 3$ into $g'(t)$ to evaluate the rate of change at that instant.

$$g'(3) = 10(3) + 2 = 32 \text{ m s}^{-1}$$

A1

Warning: The answer must include units (m s^{-1}) to fully interpret the rate of change in context.

Q3 — Gradient of a cooling curve [EASY]

Differentiate $T(x) = 20 - 6x + x^2$ term by term; the constant 20 vanishes, $-6x$ differentiates to -6, and x^2 differentiates to $2x$.

$$T'(x) = -6 + 2x$$

M1A1

Substitute $x = 4$ into $T'(x)$ to find the gradient at that point.

$$T'(4) = -6 + 2(4) = 2$$

A1

Warning: The gradient here is positive, meaning the temperature is increasing at $x = 4$; do not confuse this with the drink still cooling.

Q4 — Stationary points of a cubic [EASY]

(a) Differentiate $y = x^3 - 3x^2 + 2$ term by term using the power rule.

$$\frac{dy}{dx} = 3x^2 - 6x$$

A1A1

(b) At stationary points $\frac{dy}{dx} = 0$, so read the x-coordinates directly from the labelled diagram.
GDC: Graph $y = 3x^2 - 6x$ and find zeros, or factorise: $3x(x-2) = 0$.

$$x = 0 \quad \text{and} \quad x = 2$$

A1

Warning: The question asks only for the x-coordinates; writing the full coordinate pairs is acceptable but writing only the y-values would not earn the mark.

Q5 — Maximising profit [EASY]

Differentiate $P(n) = -2n^2 + 12n - 10$ with respect to n to find the gradient function.

$$P'(n) = -4n + 12$$

M1

Set $P'(n) = 0$ and solve for n to locate the stationary point.

$$-4n + 12 = 0 \implies n = 3$$

A1

Since $P'(n) > 0$ for $n < 3$ and $P'(n) < 0$ for $n > 3$, the function changes from increasing to decreasing, confirming $n = 3$ gives a local maximum. The profit is maximised when **n** $=$ **3** batches are baked per day.
A1
Warning: Always verify the nature of the stationary point using a sign table of P'; simply finding where $P' = 0$ without classification is incomplete.

Q6 — Tangent to a cubic curve [EASY]

Differentiate $y = 4x^3 - x$ to obtain the gradient function.

$$\frac{dy}{dx} = 12x^2 - 1$$

M1

Evaluate the gradient at $x = 1$ and find the y-coordinate of the point.

$$\left.\frac{dy}{dx}\right|_{x=1} = 12(1)^2 - 1 = 11, \qquad y = 4(1)^3 - 1 = 3$$

A1

Use $y - y_1 = m(x - x_1)$ with $m = 11$ and point $(1, 3)$ to write the tangent equation.

$$y - 3 = 11(x - 1) \implies y = 11x - 8$$

A1

Warning: Substitute $x = 1$ into the original function to find the y-coordinate; do not assume the point lies on the x-axis.

Q7 — Decreasing interval for water depth [EASY]

(a) Differentiate $D(t) = t^2 - 8t + 20$ term by term.

$$D'(t) = 2t - 8$$

A1

(b) The depth is decreasing when $D'(t) < 0$, so solve the inequality.

$$2t - 8 < 0 \implies t < 4$$

M1

Combined with the domain $0 \leq t \leq 8$, the depth is decreasing for $\mathbf{0 \leq t < 4}$ hours. **A1**

Warning: The domain restriction $0 \leq t \leq 8$ must be applied; writing $t < 4$ without the lower bound is incomplete.

Q8 — Normal to a quadratic curve [EASY]

Differentiate $y = 2x^2 - 8x + 5$ to find the gradient of the tangent at $P(3, 1)$.

$$\frac{dy}{dx} = 4x - 8 \implies \left.\frac{dy}{dx}\right|_{x=3} = 4(3) - 8 = 4$$

M1

The gradient of the normal is the negative reciprocal of the tangent gradient.

$$m_{\text{normal}} = -\frac{1}{4}$$

A1

Use $y - y_1 = m(x - x_1)$ with $m = -\frac{1}{4}$ and point $(3, 1)$.

$$y - 1 = -\frac{1}{4}(x - 3) \implies y = -\frac{1}{4}x + \frac{7}{4}$$

A1

Warning: The normal uses the negative reciprocal of the tangent gradient, not the tangent gradient itself; using $m = 4$ for the normal is the built-in trap here.

Q9 — Increasing interval of a cubic [EASY]

(a) Differentiate $f(x) = x^3 - 6x^2 + 9x + 1$ term by term.

$$f'(x) = 3x^2 - 12x + 9$$

A1

(b) The function is increasing where $f'(x) > 0$. Set $f'(x) = 0$ to find the boundary values.
GDC: Equation solver $\rightarrow 3x^2 - 12x + 9 = 0 \implies x = 1$ or $x = 3$. **M1**
Test the sign of $f'(x)$: for $x < 1$, $f'(0) = 9 > 0$; for $1 < x < 3$, $f'(2) = -3 < 0$; for $x > 3$, $f'(4) = 9 > 0$.
The function is increasing for $x < 1$ and $x > 3$. **A1**
Warning: The function is increasing on two separate intervals; writing only one of them would lose the mark.

Q10 — Particle at rest [EASY]

The velocity of the particle is the derivative of displacement with respect to time. Differentiate $s(t) = 3t^2 - 6t + 4$.

$$s'(t) = 6t - 6$$

M1

The particle is momentarily at rest when the velocity equals zero; set $s'(t) = 0$ and solve.

$$6t - 6 = 0 \implies t = 1 \text{ second}$$

A1A1

Warning: The question asks for the time, not the displacement; substituting $t = 1$ back into $s(t)$ to give a distance answer would not earn the final mark.

Q11 — Rate of change of depth model [MEDIUM]

(a) Differentiate each term of $d(t)$ using the power rule to obtain the rate of change function.

$$d'(t) = 3(0.5)t^2 - 2(6)t + 18 = 1.5t^2 - 12t + 18$$

M1 A1

(b) Substitute $t = 3$ into $d'(t)$ to find the instantaneous rate of change at that moment.

$$d'(3) = 1.5(9) - 12(3) + 18 = 13.5 - 36 + 18 = -4.5 \text{ m h}^{-1}$$

M1 A1

The units are metres per hour.
Warning: A positive sign error here (writing $+12t$ instead of $-12t$) costs the A1; always check the sign when differentiating a negative coefficient.

Q12 — Tangent to a quadratic at a point [MEDIUM]

(a) Differentiate $f(x) = 3x^2 - 8x + 5$ to get the gradient function.

$$f'(x) = 6x - 8$$

Substitute $x = 4$ to find the gradient at P.

$$f'(4) = 6(4) - 8 = 24 - 8 = 16$$

M1 A1

(b) Find the y-coordinate of the point of tangency: $f(4) = 3(16) - 8(4) + 5 = 48 - 32 + 5 = 21$. Wait — recompute: $3(16) = 48$, $8(4) = 32$, so $f(4) = 48 - 32 + 5 = 21$.
Hmm, checking the figure label $P(4, 13)$: $3(4)^2 - 8(4) + 5 = 48 - 32 + 5 = 21$. The figure should show $P(4, 21)$; use the computed value.
$f(4) = 3(16) - 32 + 5 = 21$. Use point–slope form $y - 21 = 16(x - 4)$.

$$y = 16x - 64 + 21 = 16x - 43$$

M1 A1

Warning: A common error is to use the y-value from an earlier incorrect substitution; always recompute $f(a)$ freshly before forming the tangent equation.

Q13 — Stationary points of a cubic [MEDIUM]

(a) Differentiate and set equal to zero to locate stationary points.

$$f'(x) = 3x^2 - 12x + 9 = 0 \implies x^2 - 4x + 3 = 0 \implies (x-1)(x-3) = 0$$

GDC: Equation solver or graph of $f'(x)$ to confirm roots. **M1**

$$x = 1 \quad \text{and} \quad x = 3$$

A1

(b) Use a sign table of $f'(x)$ to classify each point.
At $x = 0$: $f'(0) = 9 > 0$; at $x = 2$: $f'(2) = 12-24+9 = -3 < 0$; at $x = 4$: $f'(4) = 48-48+9 = 9 > 0$. **M1**
Since f' changes from $+$ to $-$ at $x = 1$, there is a **local maximum** at $x = 1$. Since f' changes from $-$ to $+$ at $x = 3$, there is a **local minimum** at $x = 3$. **A1**
Warning: Asserting "local maximum" or "local minimum" without showing the sign change of f' earns no marks; the classification evidence must be explicit.

Q14 — Maximum profit quadratic model [MEDIUM]

(a) Differentiate $P(x) = -2x^2 + 12x - 7$ and set $P'(x) = 0$ to find the maximising value.

$$P'(x) = -4x + 12 = 0 \implies x = 3$$

M1 A1

GDC: Graph $P(x)$ on $[0, 6]$ and use *maximum* feature to confirm $x = 3$.
(b) Substitute $x = 3$ into $P(x)$ to find the maximum profit.

$$P(3) = -2(9) + 12(3) - 7 = -18 + 36 - 7 = 11$$

M1 A1

The maximum profit is $\$11\,000$.
Warning: The units of P are thousand dollars; writing \$11 instead of \$11 000 loses the contextual accuracy of the answer.

Q15 — Differentiating a negative power [MEDIUM]

(a) Rewrite $g(x)$ using a negative index before differentiating; this is the mandatory first step.

$$g(x) = 4x^{-2} + 3x$$

Apply the power rule term by term:

$$g'(x) = -8x^{-3} + 3 = -\frac{8}{x^3} + 3$$

M1 A1

Warning: Differentiating $4x^{-2}$ as $+8x^{-3}$ (sign error on a negative power) is the most common trap here; the rule gives $n \cdot ax^{n-1} = (-2)(4)x^{-3} = -8x^{-3}$.
(b) Substitute $x = 2$ into $g'(x)$.

$$g'(2) = -\frac{8}{8} + 3 = -1 + 3 = 2$$

M1 A1

Q16 — Decreasing interval and local minimum [MEDIUM]

(a) Differentiate $h(x) = x^3 - 3x^2 - 9x + 4$ and solve $h'(x) < 0$.

$$h'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$$

M1

GDC: Graph $h'(x)$ and identify the interval where it is negative.
$h'(x) < 0$ for $-1 < x < 3$, so h is decreasing on $(-1,\, 3)$. **A1**

(b) The local minimum occurs at $x = 3$ (where f' changes from $-$ to $+$).

$$h(3) = 27 - 27 - 27 + 4 = -23$$

M1 A1

The local minimum is at $(3, -23)$.
Warning: Giving the local maximum coordinates $(-1, 9)$ instead of the local minimum when the question specifically asks for the minimum is a common mix-up; always check whether f' changes $- \to +$ (minimum) or $+ \to -$ (maximum).

Q17 — Normal to a quadratic [MEDIUM]

(b) (written first as it feeds part (a)) Write down the gradient of the tangent at $x = 3$.

$$f'(x) = 4x - 5, \quad f'(3) = 12 - 5 = 7$$

A1

(a) The gradient of the normal is the negative reciprocal of the tangent gradient.

$$m_{\text{normal}} = -\frac{1}{7}$$

M1

Find the point of contact: $f(3) = 2(9) - 15 + 1 = 18 - 15 + 1 = 4$, so the point is $(3, 4)$.
Use point–slope form:

$$y - 4 = -\tfrac{1}{7}(x - 3) \implies y = -\tfrac{1}{7}x + \tfrac{3}{7} + 4 = -\tfrac{1}{7}x + \tfrac{31}{7}$$

M1 A1

Warning: Using the tangent gradient 7 (instead of its negative reciprocal $-\frac{1}{7}$) for the normal equation is the key trap in normal-line problems; always invert *and* negate.

Q18 — Rate of change of projectile height [MEDIUM]

(a) Differentiate $h(t) = -4.9t^2 + 14.7t + 1.2$ term by term.

$$h'(t) = -9.8t + 14.7$$

A1

(b) At maximum height, $h'(t) = 0$. Solve for t.

$$-9.8t + 14.7 = 0 \implies t = \frac{14.7}{9.8} = 1.5 \text{ s}$$

M1 A1

GDC: Graph $h(t)$ and use *maximum* to confirm $t = 1.5$.

(c) $h'(t) < 0$ means the height of the ball is decreasing; the ball is falling after reaching its maximum height. **R1**

Warning: Premature rounding of $14.7 \div 9.8$ (e.g. writing 1.50 prematurely and not checking it is exact) can cause errors in subsequent parts; show the unrounded fraction $\frac{14.7}{9.8} = 1.5$ first.

Q19 — Finding points with a given gradient [MEDIUM]

(a) Differentiate $f(x) = x^3 - 12x + 2$ and set $f'(x) = 0$.

$$f'(x) = 3x^2 - 12 = 0 \implies x^2 = 4 \implies x = \pm 2$$

A1

(b) Set $f'(x) = 24$ and solve for x.

$$3x^2 - 12 = 24 \implies 3x^2 = 36 \implies x^2 = 12$$

M1

$$x = \pm\sqrt{12} = \pm 2\sqrt{3} \approx \pm 3.46$$

A1 A1

GDC: Graph $y = f'(x) = 3x^2 - 12$ and $y = 24$; use *intersection* to confirm $x \approx \pm 3.464$.

Warning: Forgetting the negative solution $x = -2\sqrt{3}$ (since $x^2 = 12$ has two roots) costs one A1; always consider both square roots when solving $x^2 = k$.

Q20 — Optimisation with AG firewall [MEDIUM]

(a) Let the length be l metres. The perimeter condition gives $2x + 2l = 40$, so $l = 20 - x$.
The area is:

$$A = x \cdot l = x(20 - x) = 20x - x^2 \quad \textbf{AG}$$

No marks awarded for the printed result; marks are for the method leading to it. **(M1)**

(b) Differentiate $A(x) = 20x - x^2$ and set $A'(x) = 0$.

$$A'(x) = 20 - 2x = 0 \implies x = 10$$

M1 A1

GDC: Graph $A(x)$ on $[0, 20]$ and use *maximum* to confirm $x = 10$.
Substitute to find the maximum area:

$$A(10) = 20(10) - 100 = 200 - 100 = 100 \text{ m}^2$$

A1

Warning: Using the AG expression from part (a) is compulsory here (the "hence" chain); deriving a different expression from scratch and getting a different answer loses all marks in part (b).

Q21 — Polynomial profit optimisation [HARD]

(a) Differentiating term by term using the power rule, the derivative of $P(x)$ gives the rate of change of profit with respect to price.

$$P'(x) = -6x^2 + 60x - 100$$

A1

(b) To find stationary points, set $P'(x) = 0$ and solve.

GDC: Enter $-6x^2 + 60x - 100 = 0$ into the equation solver (Algebra $\to$ Solve), giving $x = \dfrac{60 \pm \sqrt{3600 - 2400}}{12}$; the solutions are $x \approx 2.11$ and $x \approx 7.89$. **M1**

To classify, evaluate $P'(x)$ either side of each stationary point (sign of P'):
At $x = 2.11$: $P'(2) = -6(4)+60(2)-100 = -24+120-100 = -4 < 0$; $P'(3) = -6(9)+60(3)-100 = 26 > 0$, so local minimum at $x \approx 2.11$.
At $x = 7.89$: $P'(7) = -6(49) + 60(7) - 100 = 26 > 0$; $P'(9) = -6(81) + 60(9) - 100 = -94 < 0$, so local maximum at $x \approx 7.89$.
Within $2 \leq x \leq 20$, also check endpoints: $P(2) = -16 + 120 - 200 + 80 = -16$; $P(20) = -16000 + 12000 - 2000 + 80 = -5920$; $P(7.89) \approx P(8) = -1024 + 1920 - 800 + 80 = 176$.
The maximum profit occurs at $x \approx 7.89$, rounded to $x \approx 7.89$ (exact: $x = 5 - \frac{\sqrt{3}}{3} \cdot \sqrt{10} \approx 7.89$; accept GDC value ≈ 7.89). **A1**

R1: Since P' changes from positive to negative at $x \approx 7.89$ (confirmed by sign of P' either side), this is a local maximum, and checking endpoints confirms it is the global maximum on the domain. **R1**

Warning: A common trap is to ignore the endpoints of the restricted domain $[2, 20]$; always evaluate P at both endpoints and compare with the local maximum value.

(c) The maximum profit in thousands of dollars is $P(7.89\ldots)$; substituting the unrounded $x = 5 + \frac{\sqrt{300}}{6} = 5 + \sqrt{\frac{25}{3}}$ or using GDC directly:

GDC: evaluate P at the exact stationary value; $P_{\text{max}} \approx 176.9$ thousand dollars, i.e. **$176 900** (to nearest dollar, or accept $\approx \$177\,000$). **A1**

Warning: The function gives P in *thousands* of dollars; forgetting to multiply by 1000 loses the context mark.

Q22 — Differentiating negative powers; increasing function [HARD]

(a) Rewrite $f(x)$ using negative powers before differentiating; this is the non-negotiable first step for terms with x in the denominator.

$$f(x) = 3x^2 - 48x^{-2}$$

Differentiating term by term using the power rule ($\frac{d}{dx}[x^n] = nx^{n-1}$):

$$f'(x) = 6x - 48 \cdot (-2)x^{-3} = 6x + 96x^{-3} = 6x + \frac{96}{x^3}$$

M1

This matches the printed target. **AG**

Warning: The most frequent error is writing $\frac{d}{dx}[-48x^{-2}] = +96x^{-3}$ correctly but then stating $-96x^{-3}$ (sign error on the negative power); show the multiplication step explicitly.

(b) For all $x > 0$: both terms of $f'(x) = 6x + \dfrac{96}{x^3}$ are strictly positive since $x > 0$, so $f'(x) > 0$ for all $x > 0$. **M1**

R1: Since $f'(x) > 0$ for all x in the domain, f is strictly increasing on $x > 0$. **R1**

(c) First find $f(2)$ and $f'(2)$ to obtain the gradient and a point on the curve.

$$f(2) = 3(4) - \frac{48}{4} = 12 - 12 = 0, \quad f'(2) = 6(2) + \frac{96}{8} = 12 + 12 = 24$$

Tangent equation using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (2, 0)$ and $m = 24$:

$$y = 24(x - 2) = 24x - 48$$

A1

Q23 — Optimisation with AG firewall; domain validity [HARD]

(a) The side parallel to the wall has length $60 - 2x$ metres (since two sides of length x use $2x$ metres of fencing, leaving $60 - 2x$ for the third side). The area is:

$$A = x(60 - 2x) = 60x - 2x^2$$

AG

This is the printed target; no marks awarded for the statement alone.

(b) Differentiating $A = 60x - 2x^2$ with respect to x:

$$\frac{dA}{dx} = 60 - 4x$$

A1

(c) Set $\dfrac{dA}{dx} = 0$ to find the stationary point:

$$60 - 4x = 0 \implies x = 15$$

A1

(d) Hence (using the result $x = 15$ from part (c)):

$$A = 60(15) - 2(15)^2 = 900 - 450 = 450 \text{ m}^2$$

A1

Warning: The "hence" instruction makes it compulsory to use $x = 15$ from part (c); restarting with a different method earns no credit.

(e) When $x = 35$, the third side has length $60 - 2(35) = -10$ metres, which is negative; a length cannot be negative, so the model is not appropriate for $x = 35$. **A1**

Warning: Simply stating "it is outside the domain" is insufficient; state explicitly that the resulting side length becomes negative, which is physically impossible.

Q24 — Stationary points, normal, increasing intervals [HARD]

(a) The local maximum and local minimum are read directly from the labelled diagram (verified by $g'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$ giving $x = 1$ and $x = 3$):

$$\text{Local maximum: } (1,\, 6); \quad \text{Local minimum: } (3,\, 2)$$

A1

(b) First find $g'(x)$ and evaluate it at $x = 2$ to obtain the tangent gradient; the normal gradient is the negative reciprocal.

$$g'(x) = 3x^2 - 12x + 9$$

$$g'(2) = 3(4) - 12(2) + 9 = 12 - 24 + 9 = -3$$

M1

Gradient of normal: $m_n = -\dfrac{1}{-3} = \dfrac{1}{3}$. **M1**

Point on curve: $g(2) = 8 - 24 + 18 + 2 = 4$, so the point is $(2, 4)$.

Normal equation: $y - 4 = \dfrac{1}{3}(x-2)$, i.e. $y = \dfrac{1}{3}x + \dfrac{10}{3}$. **A1**

Warning: Using the tangent gradient (-3) instead of its negative reciprocal for the normal is the most common error; always show the negative reciprocal step explicitly.

(c) g is increasing where $g'(x) > 0$, i.e. $3(x-1)(x-3) > 0$, which holds for $x < 1$ or $x > 3$. **A1**

Q25 — Negative power rewrite; minimum of a rational model [HARD]

(a) Rewrite $T(t)$ expressing the term $\dfrac{120}{t}$ as a negative power (non-negotiable step shown explicitly):

$$T(t) = 120t^{-1} + 4t - 10$$

Differentiate term by term:

$$T'(t) = -120t^{-2} + 4 = -\frac{120}{t^2} + 4$$

M1

A1

Warning: Writing $\dfrac{d}{dt}[120t^{-1}] = +120t^{-2}$ (forgetting the negative sign from the power rule) is the flagged sign-error trap; show the step $(-1) \times 120 = -120$ explicitly.

(b) Set $T'(t) = 0$ to find the stationary point:

$$-\frac{120}{t^2} + 4 = 0 \implies \frac{120}{t^2} = 4 \implies t^2 = 30 \implies t = \sqrt{30} \approx 5.477 \text{ min}$$

M1

Sign of $T'(t)$: for $0 < t < \sqrt{30}$, $T'(t) < 0$ (decreasing); for $t > \sqrt{30}$, $T'(t) > 0$ (increasing).

R1: Since T' changes from negative to positive at $t = \sqrt{30}$, this is a local minimum. **R1**

(c) Minimum temperature (substituting $t = \sqrt{30}$, unrounded):

$$T(\sqrt{30}) = \frac{120}{\sqrt{30}} + 4\sqrt{30} - 10 = \frac{120\sqrt{30}}{30} + 4\sqrt{30} - 10 = 4\sqrt{30} + 4\sqrt{30} - 10 = 8\sqrt{30} - 10$$

$$T_{\text{min}} = 8\sqrt{30} - 10 \approx 43.8 - 10 = 33.8\,^\circ\text{C} \quad (\approx 33.8\,^\circ\text{C})$$

A1

Q26 — Gradient, tangent equation, stationary points [HARD]

(a) Differentiate $h(x) = 2x^3 - 9x^2 + 12x - 1$ term by term using the power rule:

$$h'(x) = 6x^2 - 18x + 12$$

A1

(b) Evaluate the derivative at $x = 3$ (substituting into the expression from part (a)):

$$h'(3) = 6(9) - 18(3) + 12 = 54 - 54 + 12 = 12$$

A1

(c) Find the y-coordinate: $h(3) = 2(27) - 9(9) + 12(3) - 1 = 54 - 81 + 36 - 1 = 8$, giving point $(3, 8)$.
Tangent equation using point-gradient form with $m = 12$: **M1**

$$y - 8 = 12(x - 3) \implies y = 12x - 28$$

A1

(d) Set $h'(x) = 0$:

$$6x^2 - 18x + 12 = 0 \implies x^2 - 3x + 2 = 0 \implies (x-1)(x-2) = 0$$

$$x = 1 \text{ and } x = 2$$

A1

Warning: A common error is to use the y-value from the diagram as the gradient; always compute $h'(3)$ algebraically or via GDC.

Q27 — Cubic height model; classification by sign diagram [HARD]

(a) Differentiate $H(t) = -t^3 + 9t^2 - 15t + 30$ term by term:

$$H'(t) = -3t^2 + 18t - 15$$

A1

(b) Set $H'(t) = 0$ and factorise:

$$-3t^2 + 18t - 15 = 0 \implies t^2 - 6t + 5 = 0 \implies (t-1)(t-5) = 0$$

$$t = 1 \text{ s and } t = 5 \text{ s}$$

A1

(c) Build a sign diagram for $H'(t) = -3(t-1)(t-5)$:

Interval	$t < 1$	$t = 1$	$1 < t < 5$	$t = 5$	$t > 5$
$H'(t)$	$-$	0	$+$	0	$-$
H	decreasing		increasing		decreasing

M1

R1: At $t = 1$, H' changes from negative to positive $\Rightarrow$ local **minimum**; at $t = 5$, H' changes from positive to negative $\Rightarrow$ local **maximum**. **R1**

Warning: Asserting "local maximum" or "local minimum" without evidence from the sign of H' on either side earns no marks; the sign diagram is compulsory.

(d) Compare H at the local max and both endpoints of $[0, 8]$:

$$H(5) = -125 + 225 - 75 + 30 = 55; \quad H(0) = 30; \quad H(8) = -512 + 576 - 120 + 30 = -26$$

Maximum height is **55 metres** at $t = 5$ s. **A1**

Q28 — Parameter recovery from derivative and function conditions [HARD]

(a) The question deliberately presents a false derivation to expose a common error. Find $f'(x)$ correctly first:

$$f'(x) = 2ax + b$$

Condition $f'(1) = 0$: $2a(1) + b = 0$, so $2a + b = 0$. **M1**

Condition $f(1) = 7$: $a(1)^2 + b(1) + 3 = 7$, so $a + b = 4$.
The two correct equations are $2a + b = 0$ and $a + b = 4$. **A1**

Warning: The question states an erroneous first equation "$a + b = 0$" (missing the coefficient 2 from differentiating ax^2); using this incorrect equation leads to wrong values of a and b throughout. Always differentiate before substituting.

(b) Subtract the second equation from the first to eliminate b:

$$(2a + b) - (a + b) = 0 - 4 \implies a = -4$$

M1

Substitute into $a + b = 4$: $-4 + b = 4 \implies b = 8$. **A1**

(c) The stationary point occurs where $f'(x) = 0$, i.e. at $x = 1$ (given). The y-coordinate is $f(1) = 7$ (given). Stationary point: $(1, 7)$. **A1**

Q29 — Cubic stationary points and normal at a given point [HARD]

(a) Differentiate $y = x^3 - 3x^2 - 9x + 5$ term by term:

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

A1

(b) Set $\dfrac{dy}{dx} = 0$ to find stationary points:

$$3x^2 - 6x - 9 = 0 \implies x^2 - 2x - 3 = 0 \implies (x-3)(x+1) = 0$$

M1

$x = 3$: $y = 27 - 27 - 27 + 5 = -22$. $\quad x = -1$: $y = -1 - 3 + 9 + 5 = 10$.
Stationary points: $(-1,\ 10)$ and $(3,\ -22)$. **A1**
(c) Gradient of curve at $P(-1,\ 10)$:

$$\left.\frac{dy}{dx}\right|_{x=-1} = 3(1) - 6(-1) - 9 = 3 + 6 - 9 = 0$$

M1

Since the tangent gradient is 0, the tangent is horizontal, so the normal is vertical with equation $x = -1$. **A1**

Warning: When the tangent gradient is 0, the normal gradient is undefined (vertical line), not $\dfrac{1}{0}$; the normal equation is $x = -1$, not a sloped line. Attempting $y = 10 - \frac{1}{0}(x+1)$ or similar scores zero.

Q30 — Cylinder volume optimisation with AG firewall [HARD]

(a) The total surface area of a closed cylinder is $2\pi r^2 + 2\pi rh = 300\pi$. Divide through by π:

$$2r^2 + 2rh = 300 \implies h = \frac{300 - 2r^2}{2r} = \frac{150}{r} - r$$

M1

Volume of cylinder: $V = \pi r^2 h$. Substitute the expression for h:

$$V = \pi r^2\left(\frac{150}{r} - r\right) = \pi r^2 \cdot \frac{150}{r} - \pi r^2 \cdot r = 150\pi r - \pi r^3$$

AG

Warning: The surface area formula must include *both* circular ends ($2\pi r^2$) since the cylinder is *closed*; omitting one end gives an incorrect expression for h and the AG cannot be reached.
(b) Differentiate $V = 150\pi r - \pi r^3$ with respect to r:

$$\frac{dV}{dr} = 150\pi - 3\pi r^2$$

A1

(c) Set $\dfrac{dV}{dr} = 0$ to find the stationary value of r:

$$150\pi - 3\pi r^2 = 0 \implies r^2 = 50 \implies r = \sqrt{50} = 5\sqrt{2} \approx 7.07 \text{ cm}$$

M1

Sign of $\dfrac{dV}{dr}$: for $r < 5\sqrt{2}$, $\dfrac{dV}{dr} > 0$; for $r > 5\sqrt{2}$, $\dfrac{dV}{dr} < 0$.

R1: Since $\dfrac{dV}{dr}$ changes from positive to negative at $r = 5\sqrt{2}$, this is a local maximum, so the volume is maximised at $r = 5\sqrt{2} \approx 7.07$ cm. **R1**

Warning: Only the positive root $r = 5\sqrt{2}$ is physically valid since $r > 0$; discarding the negative root must be stated explicitly.